§5.6 Substitution / change of variable

"reverse chain rule for integration"

<u>recall</u>: chain rule : $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$

<u>substitution / change of variable</u>

$\int f(u)\,du \quad u(x) \; :$

$$\boxed{\int_a^b f(u)\,du = \int_{u^{-1}(a)}^{u^{-1}(b)} f(u(x))\,\frac{du}{dx}\,dx}$$

<u>remember</u>: three things to change: limits, function, differential

<u>mnemonic</u>: $du = \frac{du}{dx}\,dx$

why does this work?

$\int f(u)\,du$, $u(x)$ set $F(u) = \int f(u)\,du$, so $F'(u) = f(u)$

$$\int f(u)\,du = F(u) \xrightarrow[\text{wrt } x]{\text{diff.}} \frac{d}{dx}\left(F(u)\right) = F'(u(x)) \cdot u'(x)$$

$$\xrightarrow[\text{wrt } x]{\text{int.}} \int f(u(x))\,u'(x)\,dx = \int f(u(x))\,\frac{du}{dx}\,dx \quad \square$$

<u>Examples</u>

$\int_{x=0}^{x=1} e^{-7x}\,dx$ set $u = -7x$ useful fact: $\frac{dx}{du} = \frac{1}{\frac{du}{dx}}$

$\frac{du}{dx} = -7$

$$\int_{u=0}^{u=-7} e^u \frac{dx}{du}\,du = \int_0^{-7} e^u \left(-\frac{1}{7}\right) du = -\frac{1}{7}\left[e^u\right]_0^{-7} = -\frac{1}{7}\left(e^{-7}-1\right)$$

more examples $\int_0^2 x^2\sqrt{x^3+1}\,dx$, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$, $\int_0^1 \frac{x}{x^2+1}\,dx$

§ 5.7 More integrals

recall: $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$ so $\int \frac{1}{x}\,dx = \ln|x| + c$

$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$

$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1}$ $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + c$

$\frac{d}{dx}(e^x) = e^x$ $\int e^x\,dx = e^x + c$

recall: $b^x = e^{x\ln(b)}$ $\int b^x\,dx = \int e^{x\ln(b)}\,dx = \frac{1}{\ln(b)} e^{x\ln(b)} + c$

(sub $u = x\ln(b)$).

Example $\int \frac{1}{9+x^2}\,dx = \frac{1}{9}\int \frac{1}{1+x^2/9}\,dx = \frac{1}{9}\int \frac{1}{1+(x/3)^2}\,dx$

substitute ~~x=~~ $u = \frac{x}{3}$ ....

$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx$ sub $u = 2x$ ....

Examples

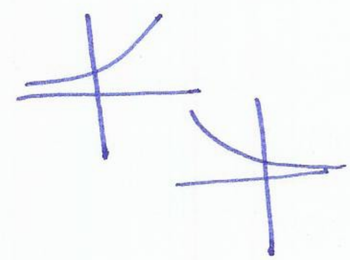
$\int x(x+1)^9\,dx \qquad \int \sqrt{4x-1}\,dx$

$\int \sin^2 x \cos x\,dx \qquad \int x\cos(x^2)\,dx \qquad \int x\sqrt{x^2-1}\,dx$

$\int \frac{1}{x\ln(x)}\,dx \qquad \int \frac{e^{2x}+e^{4x}}{e^x}\,dx \qquad \int 2^x\,dx \qquad \int xe^{x^2}\,dx$

5.8 Exponential growth and decay

$P(t) = P_0 e^{kt}$ $\qquad$ $k>0$ exponential growth
$k<0$ exponential decay



This function solves the differential equation

$\frac{dP}{dt} = kP$ $\qquad$ "rate of change of P proportional to amount of P"

Check: $\frac{dP}{dt} = P_0 k e^{kt} = kP$ $\square$

- models: population growth
  radioactive decay.
  compound interest

Doubling time / half life : length of time for P to change by factor of 2

$k>0$: doubling time $= \frac{\ln(2)}{k}$

$k<0$: half life $= \frac{\ln(2)}{-k}$

Example grow bacteria in lab, start with 1000, double every 10 mins.

$P_0 = 1000 \qquad P(t) = 1000e^{kt} \qquad k = \frac{10}{\ln(2)}.$

Interest

compound yearly: $P_0 + rP_0 = P_0(1+r)$ 1 year. $r = \frac{\text{interest rate}}{100}$

2 years: $\cancel{P_0 + rP_0}\ P_0(1+r) + rP_0(1+r) = P_0(1+r)^2$

t years: $P_0(1+r)^{t-1} + rP_0(1+r)^{t-1} = P_0(1+r)^t$

compound quarterly: (1 year): $P_0 + \frac{r}{4}P_0 = P_0\left(1+\frac{r}{4}\right)$ 1 quarter

$P_0\left(1+\frac{r}{4}\right)^2$ 2 quarters

$\vdots$

$P_0\left(1+\frac{r}{4}\right)^4$ 4 quarters.

t years: $P_0\left(1+\frac{r}{4}\right)^{4t}$

compound monthly: $P_0\left(1+\frac{r}{12}\right)^{12t}$ ($\cancel{t}$ - year)

daily: $P_0\left(1+\frac{r}{365}\right)^{365t}$

compound M times a year $P_0\left(1+\frac{r}{M}\right)^{Mt}$

$$\lim_{M\to\infty}\left(1+\frac{r}{M}\right)^M = \lim_{M\to\infty} e^{M\cancel{\log}(1+\frac{r}{M})}.$$

consider $\lim_{M\to\infty} M \ln\left(1+\frac{r}{M}\right) = \lim_{M\to\infty}\frac{\ln(1+r/M)}{1/M} \underset{\text{l'H.}}{=} \lim_{M\to\infty}\frac{\frac{1}{1+r/M}\cdot rM^{-2}\cdot -1}{-M^{-2}}$

$= \lim_{M\to\infty}\frac{r}{1+\frac{r}{M}} = r$, so $\lim_{M\to\infty} e^{M\ln(1+\frac{r}{M})} = \cancel{\lim}\ e^r\ \cancel{\text{etc}}.$

so continuously compounded interest given by P_0e^{rt}