<u>Proof</u> suppose $F(x)$ and $G(x)$ are anti-derivatives for $f(x)$.

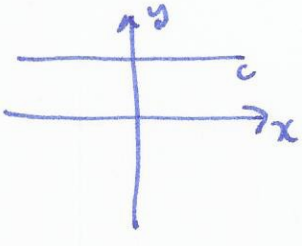
i.e. $F'(x) = f(x)$ and $G'(x) = f(x)$

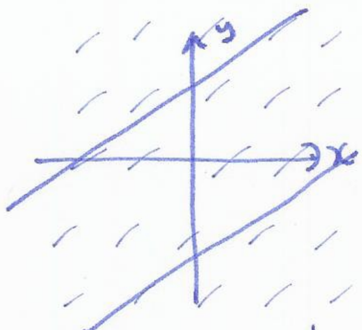
consider $G(x) - F(x)$, which has derivative $(G(x)-F(x))' = f(x) - f(x) = 0$

so $G(x) - F(x)$ is constant function. $\square$.

<u>Picture</u> $f(x)$ gives the slope function for $F(x)$

<u>Examples</u> $f(x) = c$

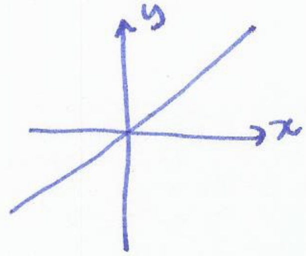


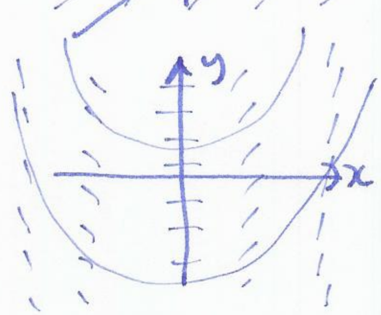


slope $F'(x) = c$ everywhere

$F(x) = cx + d$.

$f(x) = x$





$F(x) = \frac{1}{2}x^2 + c$

<u>Examples</u> : find the general anti-derivative to $f(x) = \sin(4x)$

guess : $\frac{d}{dx}\left(\cos(4x)\right) = -\sin(4x) \cdot 4$

so $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = -\frac{1}{4}\cdot(-\sin(4x))4 = \sin(4x)$

so $F(x) = -\frac{1}{4}\cos(4x) + c$

<u>Notation</u> : indefinite integral

$\int f(x)\,dx = F(x) + c$ means: $F(x) + c$ is the general anti-derivative for $f(x)$.

<u>Thm</u> $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$

<u>Proof</u> $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + c\right) = \frac{1}{n+1}(n+1)x^n = x^n$

Thm $\int \frac{1}{x}\,dx = \ln|x| + c$

Proof $(x>0)$ $\frac{d}{dx}\left(\ln(x)+c\right) = \frac{1}{x}$ $\square$

Thm sums and constant multiples:

$$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c f(x)\,dx = c\int f(x)\,dx$$

Warning: no product / quotient / chain rule!

Useful integrals

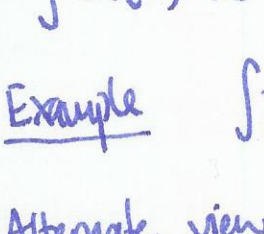
$\int \sin(x)\,dx = -\cos(x)+c$ $\qquad$ $\int e^x\,dx = e^x + c$

$\int \cos(x)\,dx = \sin(x)+c$

Example $\int x^2 + \frac{1}{x} + \sin(x)\,dx = \frac{1}{3}x^3 + \ln|x| - \cos(x) + c$

Alternate view

We can think of finding the indefinite integral as finding a function given its slope function, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general there is a family of solutions $F(x)+c$, but if we know the value of the solution we want at $x=0$ (sometimes called an <u>initial condition</u>) this gives a particular solution.



$\frac{dy}{dx} = 1$

Example an object falls freely under gravity, so

acceleration: $a(t) = -g$ (constant)

velocity : $v(t)$, has $v'(t) = a(t)$

$\frac{dv}{dt} = -g$ has general solution $v(t) = -gt + c$

if the velocity at time $t=0$ is v_0, then $v(0) = v_0 = c$

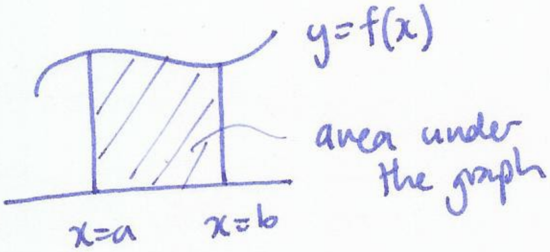
and so the particular solution is $v(t) = -gt + v_0$

position: $x(t)$, has $\frac{dx}{dt} = v(t)$

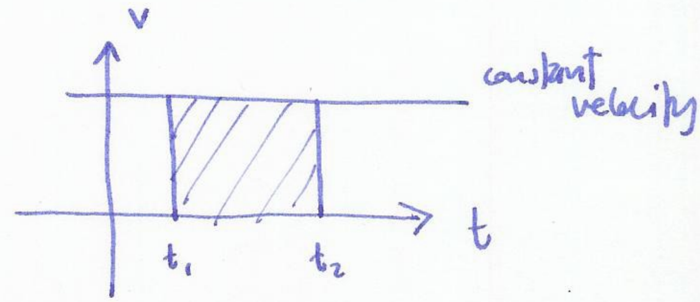
general sol$\underline{n}$ $x(t) = -\frac{1}{2}gt^2 + v_0 t + c$

if position at time $t=0$ is x_0, then $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$

§5.1 Approximating areas

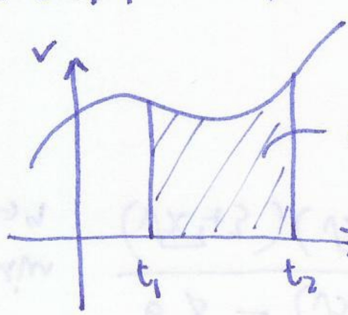


Example



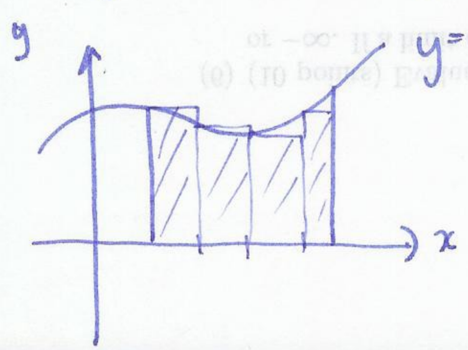
distance travelled = velocity × time
= area under the graph

non-constant velocity

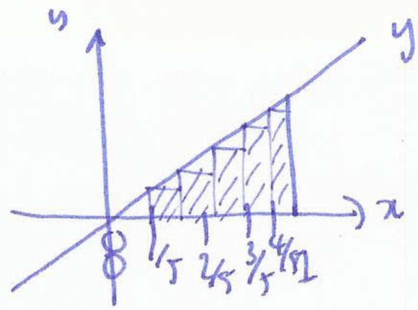


distance travelled = area under graph

finding the area: approximate by rectangles



Example



find area under $y = x$ between 0 and 1

(answer $= \frac{1}{2}$)

approximate with 5 rectangles : area $\approx$ sum of area of rectangles $=$ width $\times$ height

$$= \frac{1}{5} f(0) + \frac{1}{5} f\left(\frac{1}{5}\right) + \frac{1}{5} f\left(\frac{2}{5}\right) + \frac{1}{5} f\left(\frac{3}{5}\right) + \frac{1}{5} f\left(\frac{4}{5}\right) = \sum_{i=0}^{4} \frac{1}{5} f\left(\frac{i}{5}\right).$$

$$= \frac{1}{5}\left(0 + \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) = \frac{1}{25} \cdot 10 = \frac{10}{25} = 0.4$$

approximate with n rectangles:

$$= \frac{1}{n} f(0) + \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n-1}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right)$$

$$= \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n-1}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n} \cdot \frac{i}{n}$$

$$= \frac{1}{n^2}\left(1 + 2 + 3 + \cdots + n-1\right) = \frac{1}{n^2} \sum_{i=0}^{n-1} i$$

claim: $1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1)$

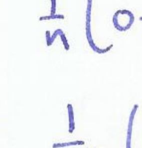
Proof (1) induction: assume true for k:

$$s_k = 1 + 2 + \cdots + k = \frac{1}{2} k(k+1)$$

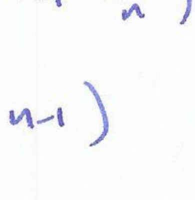
$$s_{k+1} = \underbrace{1 + 2 + \cdots + k}_{\frac{1}{2}k(k+1)} + k+1 = \frac{1}{2} k(k+1) + (k+1)$$

$$= (k+1)\left(\frac{1}{2}k + 1\right) = \frac{1}{2}(k+1)(k+2) \checkmark$$

(2)



$\frac{1}{2}(2)^2 + \frac{1}{2} \cdot 2$



$\frac{1}{2}(3)^2 + \frac{1}{2}(3)$

$\frac{1}{2}(n)^2 + \frac{1}{2} n$

$= \frac{1}{2} n(n+1)$ □

so approximate area is

$$\frac{1}{n^2}(1 + 2 + \cdots + n-1) = \frac{1}{n^2} \frac{1}{2}(n-1)n = \frac{1}{2} \frac{n-1}{n} = \frac{1}{2}\left(1 - 1/n\right) \to \frac{1}{2} \text{ as } n \to \infty.$$