<u>Example</u> sketch graph of $\dfrac{x}{\sqrt{x^2+1}} = x\,(x^2+1)^{-1/2}$

① vertical asymptotes: none.

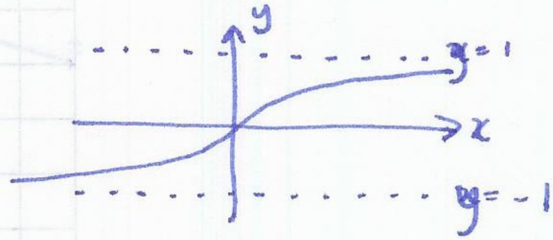
② find $f'(x) = \dfrac{\sqrt{x^2+1} - x\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x}{x^2+1} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}} > 0$ increasing, no critical points

③ find $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2}\cdot 2x$ : point of inflection at $x=0$
+ve for $x<0$
−ve for $x>0$

④ horizontal asymptotes

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x} = \lim_{x\to\infty} \sqrt{1+1/x^2} = 1$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{-x}{\sqrt{x^2+1}} = -1$$

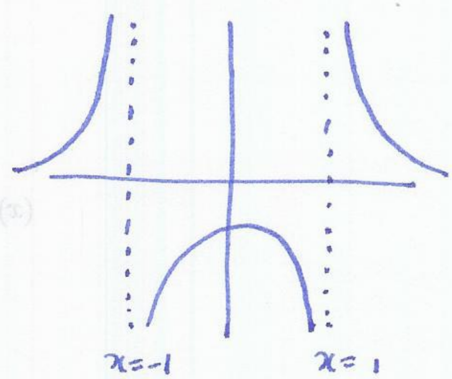



<u>Example</u> $f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)}$

① vertical asymptotes at $x = \pm 1$

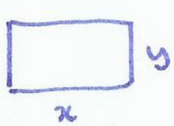
② $f'(x) = -(x^2-1)^{-2}\cdot 2x$ critical point at $x=0$

③ $f''(x) = \dfrac{6x^2+2}{(x^2-1)^3}$ etc.




<u>§4.6 Optimization</u>

<u>Example</u> A piece of wire of length L is bent into a rectangle. What's the largest possible area?

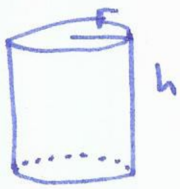



area $A = xy$
perimeter $L = 2x+2y$ $\Bigg\}$ $\Rightarrow$ $y = \frac{L}{2} - x$, $A = x\left(\frac{L}{2} - x\right)$
$= \frac{Lx}{2} - x^2$.

$A'(x) = \frac{L}{2} - 2x$ critical point $x = \frac{L}{4}$ (max)

so max area occurs at $x = \frac{L}{4} = y$ (square).

<u>Example</u> What shape of cylindrical can minimizes surface area, if you want total volume to be 1 ft³.




$$V = \cancel{4}\pi r^2 h = 1 \Leftrightarrow h = \frac{1}{\pi r^2}$$

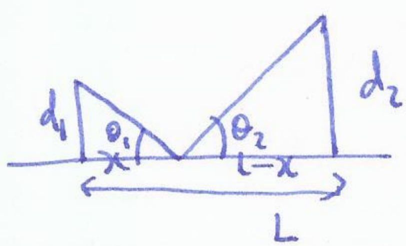
$$A = 2\pi r^2 + 2\pi r h$$

$$A = 2\pi r^2 + \frac{2\pi r}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$$

$$A'(r) = 4\pi r - \frac{2}{r^2} \quad \text{solve } A'(r) = 0 \Rightarrow r^3 = \frac{1}{2\pi}$$

so $r = \sqrt[3]{1/2\pi}$.

<u>Example</u> a ball bounces off a wall. Show that the path which minimizes distance has equal angle of reflection and incidence.




$$f(x) = \text{distance} = \sqrt{d_1^2 + x^2} + \sqrt{d_2^2 + (L-x)^2}$$

$$f'(x) = \frac{1}{2}\left(d_1^2 + x^2\right)^{-1/2} \cdot 2x + \frac{1}{2}\left(d_2^2 + (L-x)^2\right)^{-1/2} \cdot 2(L-x) \cdot -1$$

solve $f'(x) = 0$ :

$$\frac{x}{\sqrt{d_1^2 + x^2}} = \frac{L-x}{\sqrt{d_2^2 + (L-x)^2}} \Leftrightarrow \cos\theta_1 = \cos\theta_2$$

$$\Rightarrow \theta_1 = \theta_2 .$$

## <u>§4.9 Antiderivatives</u>

<u>Defn</u> A function $F(x)$ is an <u>antiderivative</u> of $f(x)$ if $F'(x) = f(x)$

<u>Example</u> $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$. check $\frac{d}{dx}(F(x)) = \frac{1}{3}3x^2 = x^2 = f(x)$ ✓

<u>Note</u> $F(x) = \frac{1}{3}x^3 + 4$ is also an antiderivative: $\frac{d}{dx}\left(\frac{1}{3}x^3 + 4\right) = x^2$.

<u>General anti-derivative</u>

<u>Thm</u> Let $F(x)$ be an anti-derivative for $f(x)$. Then any other anti-derivative for $f(x)$ is of the form $F(x) + c$ for some constant $c$.