⑤ $$\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = 0$$

⑥ $$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\ln(x)}$$: note e^x continuous so

$$e^{\lim_{x\to 0+} x\ln(x)} = \lim_{x\to 0+} e^{x\ln(x)}$$

$$\lim_{x\to 0+} x\ln x = 0 \Rightarrow \lim_{x\to 0^+} x^x = e^0 = 1$$

Comparing growth rates of functions

Q: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln(x))^2} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{2\ln(x)\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{4/x}$$

$$= \lim_{x\to\infty} \frac{1}{8}x^{1/2} = \infty.$$ so $\sqrt{x}$ grows faster.

Thm e^x grows faster than any polynomial.

Proof $$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty.$$

§4.6 Graph sketching and asymptotes

Example sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3 = f(x)$

helpful info: · critical points
· sign of $f'(x)$
· sign of $f''(x)$

$f'(x) = x^2 - x - 2 = (x+1)(x-2)$

$f''(x) = 2x-1$

critical points at $x = -1, +2$;

f': $+$ | $-$ | $+$ (at -1, 2)

$(x+1)$: $-$ | -1 | $+$

$(x-2)$: $-$ | 2 | $+$

Example $f(x) = (4x - x^2)e^x$

$f'(x) = (4+2x-x^2)e^x$

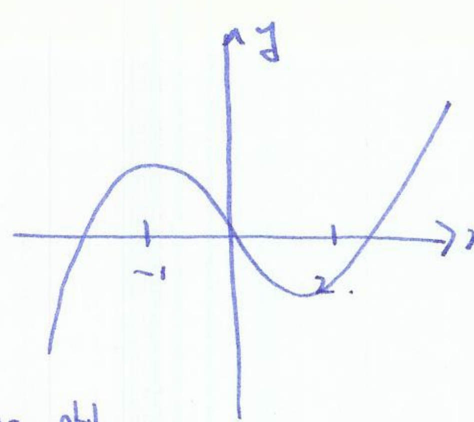
critical points $x = 1 \pm \sqrt{5}$

$f''(x) = (6-x^2)e^x$

$1+\sqrt{5}$ local max $1-\sqrt{5}$ local min

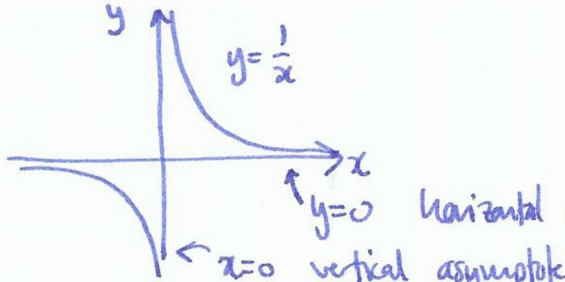
$f''(-1) = -3 < 0$ max

$f''(2) = 3 > 0$ min



Asymptotes

Example $y = \frac{1}{x}$



<u>Def^n</u> $y=c$ is a <u>horizontal asymptote</u> if $\lim_{x\to\pm\infty} f(x) = c$

$x=c$ is a <u>vertical asymptote</u> if $\lim_{x\to \pm c} f(x) = \pm\infty$

<u>Observation</u> rational functions $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \leq \deg q$.

<u>Example</u> $f(x) = \dfrac{x^2+x+1}{3x^2+2} = \dfrac{1+1/x+1/x^2}{3+2/x^2} \longrightarrow \frac{1}{3}$ as $x \to \infty$.

<u>Example</u> sketch graph of $\frac{3x+2}{2x-4} = f(x)$.

① find vertical asymptotes (when denominator is zero) $2x-4=0 \Rightarrow x=2$

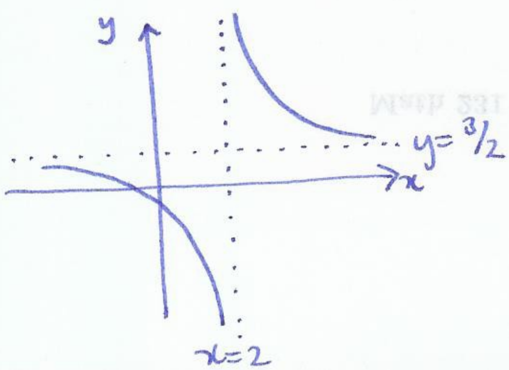
② find $f'(x) = \dfrac{(2x-4).3-(3x+2).2}{(2x-4)^2} = \dfrac{-16}{(2x-4)^2} = \dfrac{-4}{(x-2)^2}$ ← always $-$ve decreasing

no critical points!

③ $f''(x) = \dfrac{8}{(x-2)^3}$ +ve $x>2$ concave up

$-$ve $x<2$ concave down

④ horizontal asymptote $f(x) = \dfrac{3+2/x}{2-4/x}$ $\lim_{x\to\pm\infty} f(x) = \frac{3}{2}$



behaviour near asymptotes