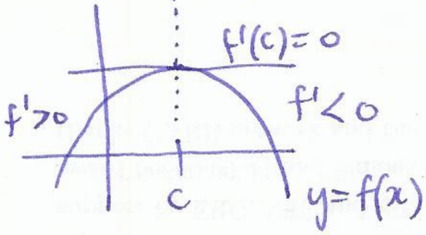
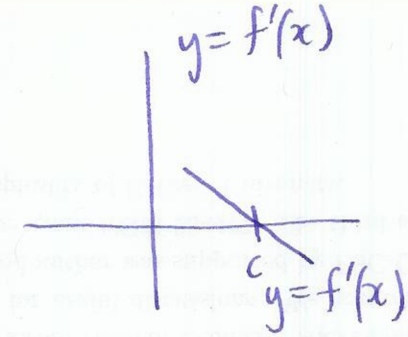
First derivative test

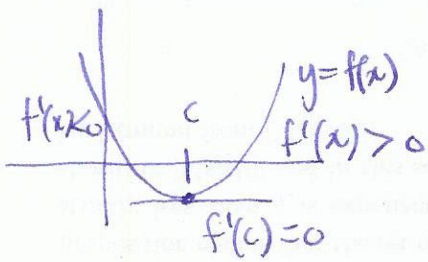
local max:

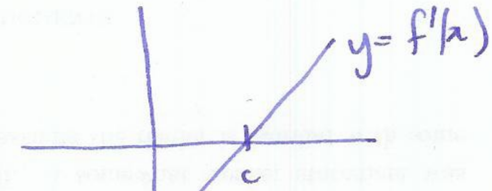




f'(x) goes from +ve to −ve ⇒ local max

local min:





f'(x) goes from −ve to +ve ⇒ local min

Thm: First derivative test

If $f(x)$ differentiable and $f'(c)=0$ (i.e. c is a critical point)

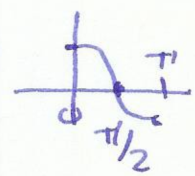
then if $f'(x)$ changes from +ve to −ve at c ⇒ c local max

−ve   +ve at c ⇒ c local min

Example classify critical points of $f(x) = \cos^2 x + \sin x$ on $[0, \pi]$

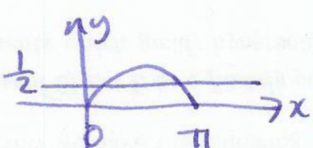
find critical points: $f'(x) = -2\cos(x)\sin(x) + \cos(x)$

solve: $f'(x)=0$ $\cos(x)\left[1 - 2\sin(x)\right] = 0$ $\cos(x)=0$



$\Rightarrow x = \frac{\pi}{2}$

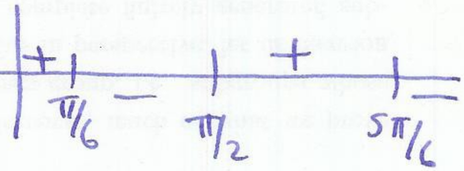
$1 - 2\sin x = 0 \Leftrightarrow \sin x = \frac{1}{2}$



$x = \frac{\pi}{6}, \frac{5\pi}{6}$

so critical points are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{2}$

find sign of $f'(x)$ between these



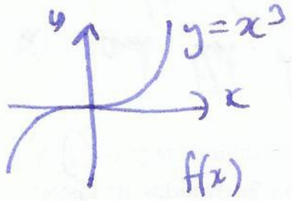
⇒ local max: $\pi/6$, $\frac{5\pi}{6}$

local min: $\pi/2$.

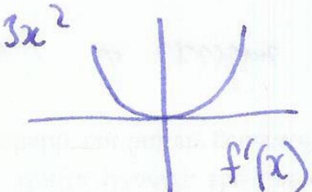
<u>Example</u> critical point <u>not</u> local max or min

54

$f(x) = x^3$



$f'(x) = 3x^2$



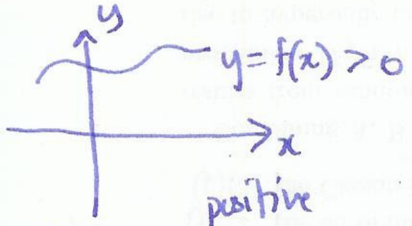
$f'(0) = 0$

+ | + f'(x)
0

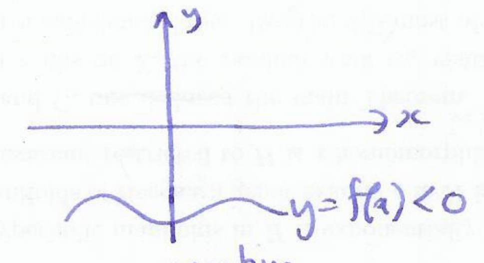
<u>§4.4 Second derivative test</u>

+ | + ⇒ <u>not</u> local max or min.
0

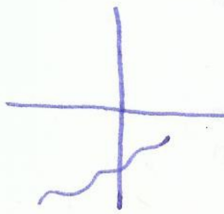
<u>recall</u>



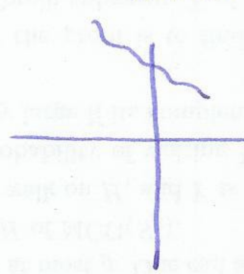
$f(x)>0$



negative.
$f(x)<0$



$f'(x)>0$
increasing

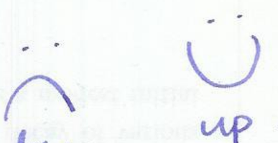


$f'(x)<0$
decreasing

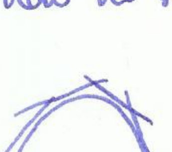
concave down.

mnemonic

Concave up



<u>Q</u>: how do the slopes change?



concave down

concave up

⟺ slopes decreasing

⟺ $f''(x) < 0$

⟺ slopes increasing

⟺ $f''(x) > 0$

<u>Def</u>

Def^n Let $f(x)$ be differentiable on the interval (a,b) then

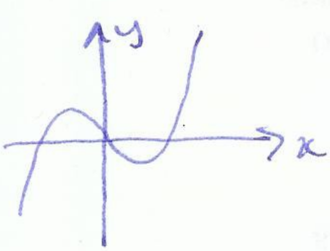
$f(x)$ is concave up $\Leftrightarrow$ $f''(x) > 0$

$f(x)$ is concave down $\Leftrightarrow$ $f''(x) < 0$

Def^n A <u>point of inflection</u> is where the graph changes from concave up to concave down (or vice versa)

Note x point of inflection $\underset{\not\Leftarrow}{\Rightarrow}$ $f''(x) = 0$

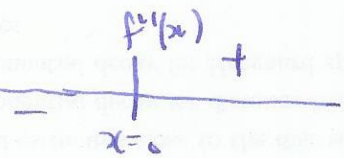
Example $f(x) = x^3 - x = x(x^2-1) = x(x-1)(x+1)$



$f'(x) = 3x^2 - 1$

$f''(x) = 6x$ $\qquad$ $f''(x) > 0$ for $x > 0$

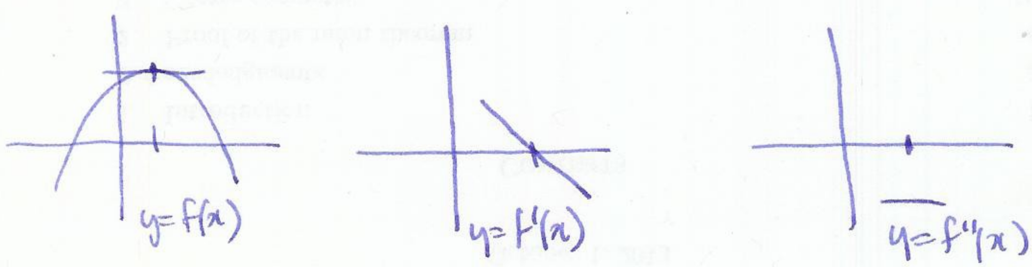
$\qquad\qquad\qquad\qquad$ $f''(x) < 0$ for $x < 0$



inflection point at $x = 0$.

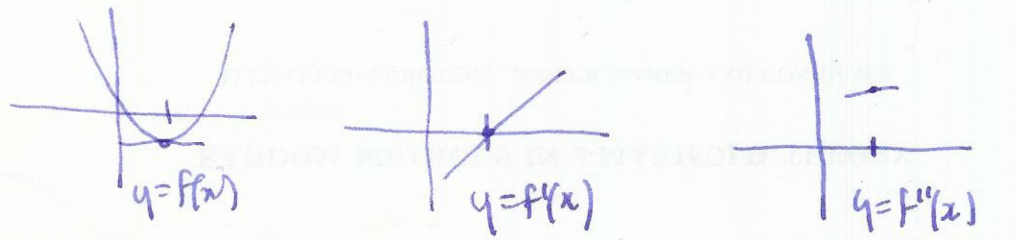
Second derivative test

local max



$f''(x) < 0$

local min



$f''(x) > 0$

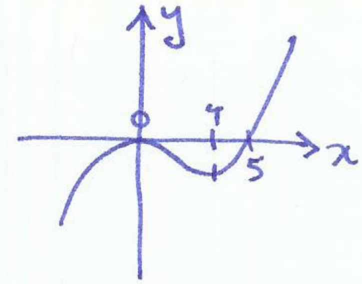
Thm suppose $f(x)$ differentiable and c is a critical point, then

if $f''(c) > 0 \Rightarrow$ ~~f(c)~~ c is a local max

$f''(c) < 0 \Rightarrow$ c is a local min

$f''(c) = 0 \Rightarrow$ NO INFORMATION! may be local max/min/neither.

Example $f(x) = x^5 - 5x^4 = x^4(x-5)$



$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$

$f''(x) = 20x^3 - 60x^2$

critical points : $f'(x) = 0 \Rightarrow x = 0, 4$

2nd derivative test: $x = 0$ $f''(0) = 0$ no information

$x = 4$ $f''(4) = 320 > 0 \Rightarrow$ local minimum.

at $x=0$ use first derivative test :

		0		4	
x^3	$-$	0	$+$		
$(x-4)$			$-$	4	$+$
$x^3(x-4)$	$+$	|	$-$	|	$+$

$\Rightarrow$ local max.

§4.5 L'Hôpitals rule

Thm suppose $f(x)$ and $g(x)$ are differentiable and $f(a) = g(a) = 0$

(or $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} g(x) = \pm\infty$) and $g'(a) \neq 0$ near a.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided this limit exists.

Warning ① this is not the quotient rule!

② $\frac{f(a)}{g(a)}$ must be in indeterminate form, can't use this on $\lim_{x \to 0} \frac{1}{x}$.

Examples ① $\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} \frac{3x^2}{1} = 3.$

② $\lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} = \lim_{x \to \frac{\pi}{2}} \frac{-2\cos x \cdot \sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}} -2\sin x = -2$

③ $\lim_{x \to 0+} x\ln(x) = \lim_{x \to 0+} \frac{\ln(x)}{1/x} = \lim_{x \to 0+} \frac{1/x}{-x^{-2}} = \lim_{x \to 0+} -x = 0$

④ $\lim_{x \to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{e^x - 1}{-\sin x} = \lim_{x \to 0} \frac{e^x}{-\cos x} = -e$