Example $y = \sinh^{-1}(x)$

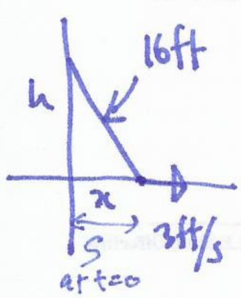
$$\sinh(y) = x$$

$$\cosh(y)\,y' = 1$$

$$y' = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{1+x^2}}$$

§3.11 Related rates

Example falling ladders



$x(t)$ distance of foot of ladder from wall

$h(t)$ height of top of ladder against wall

Q: if $\frac{dx}{dt} = 3\text{ft/s}$ what is $\frac{dh}{dt}$?

note true for all t | true at $t=0$

$$x^2 + h^2 = 16 \qquad x(0) = 5$$

$$\frac{dx}{dt} = 3$$

consider $x^2 + h^2 = 16 \Leftrightarrow (x(t))^2 + (h(t))^2 = 16$

differentiate wrt t :

$$2x(t)\underbrace{\frac{dx}{dt}}_{3} + 2h(t)\frac{dh}{dt} = 0$$

$\frac{dx}{dt} = 3$ for all $t \Rightarrow x(t) = 5 + 3t$

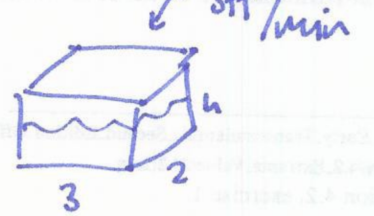
$$h(t) = \sqrt{16^2 - (5+3t)^2}$$

so $\frac{dh}{dt} = \frac{-x(t).3}{h(t)} = \frac{-3(5+3t)}{\sqrt{16^2 - (5+3t)^2}}$

hit ground at $5+3t=16 \Rightarrow t = 11/3$

$$\frac{dh}{dt}\left(11/3\right) = -\frac{16}{\sqrt{16^2-16^2}} \quad \text{i.e.} \quad \frac{dh}{dt}(t) \to \infty \text{ as } t \to \frac{11}{3}.$$

<u>Example</u> filling a rectangular tank



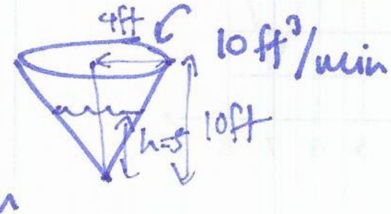
<u>Q</u>: how fast is water rising?

$V(t)$ = vol of water

$h(t)$ = height

$$V = 6h \qquad \frac{dV}{dt} = 6\frac{dh}{dt} \qquad \text{so} \quad \frac{dV}{dt} = 3 = 6\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{2} \text{ ft/min}.$$

<u>Example</u> filling a conical tank



water in at $10\text{ ft}^3/\text{min} \Leftrightarrow \frac{dV}{dt} = 10 \text{ ft}^3/\text{min}$

volume of cone $V = \frac{1}{3}\pi h r^2$

need r in terms of h: $\frac{r}{h} = \frac{4}{10} = \frac{2}{5} \Rightarrow r = \frac{2}{5}h$

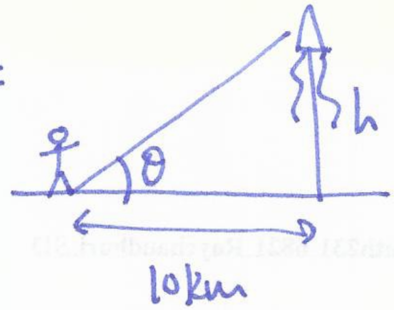
$$\text{so} \quad V = \frac{1}{3}\pi h\left(\frac{2}{5}h\right)^2 = \frac{4}{75}\pi h^3$$

$$\frac{dV}{dt} = \frac{12}{75}\pi h^2 \frac{dh}{dt}, \text{ so when } h=5 \text{ water rising at rate}$$

$$10 = \frac{12}{75}\pi(5)^2\frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{10}{4\pi} \text{ ft/min}$$

<u>advice</u>:
1. give things names (height h, volume V)
2. write down relations between things and use implicit differentiation
3. plug in numbers if necessary

Example

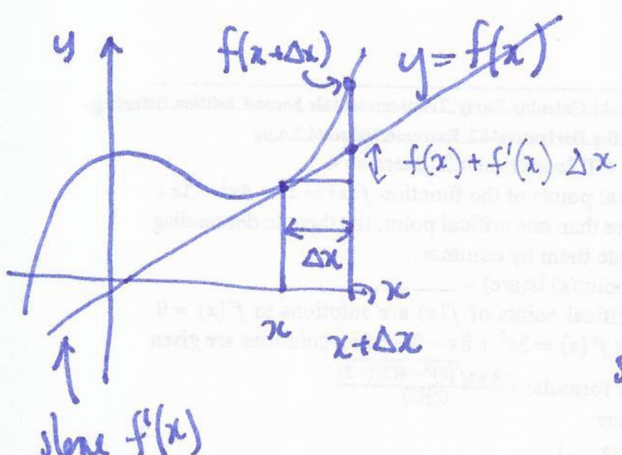


if angle is $\theta = \frac{\pi}{3}$ and rate of change $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min how fast is the rocket going?

$$\frac{h}{10} = \tan\theta \qquad \frac{1}{10}\frac{dh}{dt} = \sec^2\theta \frac{d\theta}{dt}$$

$$\frac{dh}{dt} = 10\sec^2\left(\frac{\pi}{3}\right)\cdot\frac{1}{2} \approx 20 \text{ km/min}$$

§4.1 Linear approximation



If $f(x)$ is differentiable at x and Δx is small then

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x.$$

so change in f is

$$\Delta f = f(x+\Delta x) - f(x)$$

$$\Delta f \approx f'(x)\Delta x$$

Example estimate $\sqrt{103}$

$$f(x) = \sqrt{x} = x^{1/2} \qquad f(100) = 10$$

$$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(100) = \frac{1}{20}$$

so $\Delta f \approx f'(x)\Delta x = \frac{1}{20}\cdot 3$

so $\sqrt{103} \approx 10 + \frac{3}{20} = 10.15$