Alternate notation $f(g(x)) \leftrightarrow f(u) \quad u = g(x)$

$$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$$

mnemonic "cancelling fractions"

Examples $\cos(x^2)$, $e^{\sqrt{x}}$, $\sin\left(\frac{\pi x}{180}\right)$, $\sqrt{x+\sqrt{x^2+1}}$.

Proof (of chain rule)

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

[answer should be $f'(g(x))\,g'(x)$]

write this as:

$$\lim_{h\to 0} \underbrace{\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}} \underbrace{\frac{g(x+h)-g(x)}{h}}_{\text{this gives } g'(x)} \quad (*)$$

set $k = g(x+h) - g(x)$

as g is continuous $h \to 0 \Rightarrow k \to 0$

so
$$\lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} = f'(g(x))$$

So $(*) = f'(g(x))\,g'(x)$, as required □.

More examples
$$\frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1} \cdot g'(x)$$

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x)$$

$$\frac{d}{dx}\left(f(ax+b)\right) = a f'(ax+b)$$

§1.5 Inverse functions

recall $f: X \to Y$ (domain, range), $x \mapsto f(x)$

want: the inverse function should be the reverse of this $X \xleftarrow{f^{-1}} Y$, $x \leftarrow f(x)$

problem: the inverse is often not a function

$a \xrightarrow{f} f(a) = f(b)$, $b \xrightarrow{f} f(a) = f(b)$

suppose there is $a \neq b$ s.t. $f(a) = f(b)$

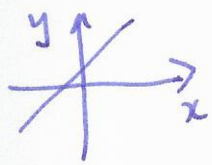
what is $f^{-1}(f(a))$?

Q: when does a function have an inverse?

A: when it's one-to-one / injective / passes the horizontal line test

$\Leftrightarrow$ for each value $c \in$ range, there is a unique $x \in$ domain s.t. $f(x) = c$

Example $y = x+1$



Q: how do we find a formula for the inverse?

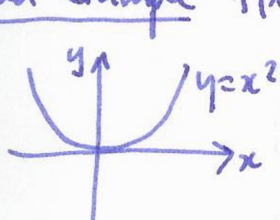
A: ① write down $y = f(x)$

② solve for x in terms of y, i.e. $x = g(y)$

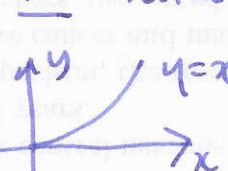
③ $f^{-1}(x) = g(x)$

④ check!

bad example $f(x) = x^2$ problem: doesn't pass horizontal line test



fix: restrict domain, consider $f: [0,\infty) \to [0,\infty)$, $x \mapsto x^2$

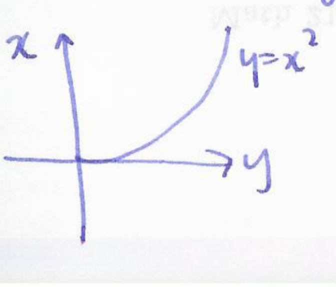


this does pass the horizontal line test, so there is an inverse we call $\sqrt{x}$.

$f^{-1}: [0,\infty) \to [0,\infty)$

$x \mapsto \sqrt{x}$

How to draw the graph of the inverse:



graph of $f(x)$: set of points $(x, f(x))$ ← $\begin{cases} x \in \text{domain of } f \\ y \in \text{range of } f \end{cases}$

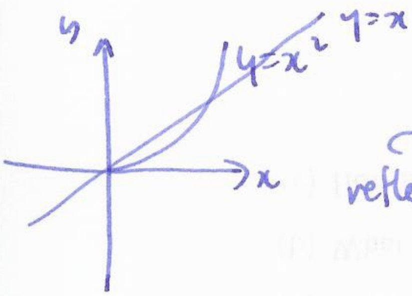
graph of $f^{-1}(x)$: set of points $(x, f^{-1}(x))$ ← $x \in$ domain of f^{-1} range of f.

normally call points in range of f y so want $(y, f^{-1}(y))$

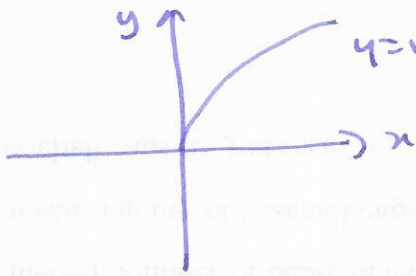
but if $y = f(x)$ then $f^{-1}(y) = x$ so want $(y, x) \Leftrightarrow (f(x), x)$

$\Leftrightarrow (y, f^{-1}(y))$

summary $\boxed{\text{graph of } f^{-1}(x) \text{ is graph of } f(x) \text{ reflected in } y = x}$

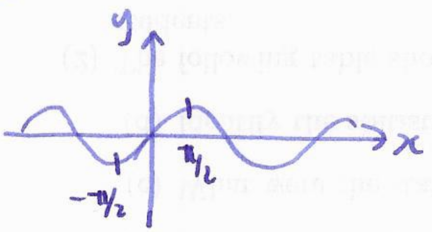


reflect



Inverse trig functions

$y = \sin(x)$

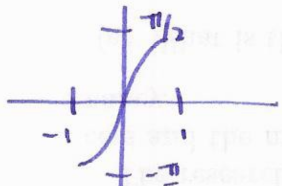


problem: not one-to-one

fix: restrict domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\sin^{-1}(x) : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



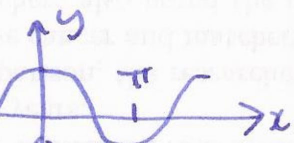
$\shortparallel$

asin(x)

$\shortparallel$

arcsin(x)

similarly $y = \cos(x)$



restrict domain to $[0, \pi]$

$\cos(x) : [0, \pi] \to [-1, 1]$

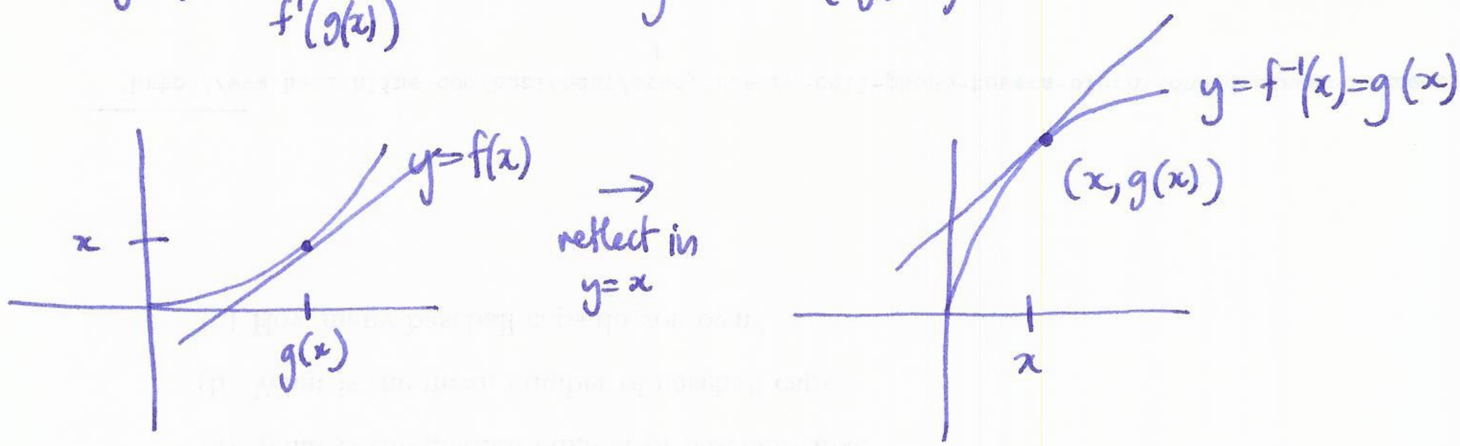
$\cos^{-1}(x) : [-1, 1] \to [0, \pi]$

§3.8 Derivatives of inverse functions

Thm Suppose $f(x)$ differentiable, one-to-one, with inverse $f^{-1}(x) = g(x)$.

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

Recall



$\dfrac{1}{\text{slope}}$ at $g(x)$ $\longleftrightarrow$ slope at x: $g'(x)$

i.e. $\dfrac{1}{f'(g(x))}$

Example $f(x) = x^2$

$g(x) = f^{-1}(x) = \sqrt{x}$

$f'(x) = 2x \qquad g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} x^{-1/2}$

Thm $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}} \qquad \dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$

Proof (of $\sin^{-1}(x)$).

$\cancel{y = \sin^{-1}(x) \quad f^{-1}(x) = \sin(x) = g(x)}$

$f(x) = \sin(x) \qquad f^{-1}(x) = \sin^{-1}(x) = g(x) \qquad$ use $g'(x) = \dfrac{1}{f'(g(x))}$.

$f'(x) = \cos(x)$

so $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\cos(\sin^{-1}(x))}$