Thm **Product rule**

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

Warning $(fg)' \neq f'g'$!!

Examples ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x).x + x.\frac{d}{dx}(x) = 1.x + x.1 = 2x$

② $$\frac{d}{dx}\left(3x^2(x^2+1)\right) = \frac{d}{dx}(3x^2).(x^2+1) + 3x^2\frac{d}{dx}(x^2+1)$$
$$= 6x(x^2+1) + 3x^2.2x$$

③ $$\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2).e^x + x^2.\frac{d}{dx}(e^x)$$
$$= 2xe^x + x^2e^x$$

Proof (of product rule) (assume f, g both differentiable at x)

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)g'(x) + g(x)f'(x) \quad \square$$

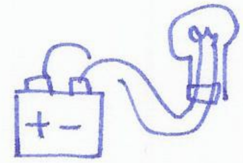
Thm Quotient rule (assume f, g differentiable at x, $g(x) \neq 0$)

Then
$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - g'(x) f(x)}{(g(x))^2}$$

Example ①
$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - (x+1)' x}{(x+1)^2}$$
$$= \frac{(x+1) \cdot 1 - x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

②
$$\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t)(e^t)' - (e^t+t)'(e^t)}{(e^t+t)^2}$$
$$= \frac{(e^t+t)e^t - (e^t+1)e^t}{(e^t+t)^2} = \frac{te^t - e^t}{(e^t+t)^2}$$

③ battery power

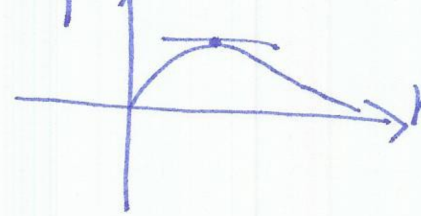


R resistance

power $P = \dfrac{V^2 R}{(r+R)^2}$

internal resistance r

Q: when does the battery give maximum power?



A: when $\dfrac{dP}{dr} = 0$

$P = \dfrac{V^2 R}{(r+R)^2}$ $\quad P(R)$; V, r constant

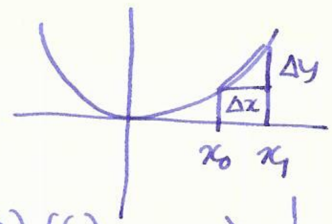
$$\frac{dP}{dR} = \frac{(r+R)^2 (V^2 R)' - V^2 R \left[(r+R)^2\right]'}{(r+R)^4} = \frac{(r+R)^2 V^2 - V^2 R (r^2 + 2rR + R^2)'}{(r+R)^4}$$

$$= \frac{V^2\left[(r+R)^2 - R(2r+2R) \right]}{(r+R)^4} = \frac{V^2 (r+R)\left[r+R - 2R \right]}{(r+R)^4}$$

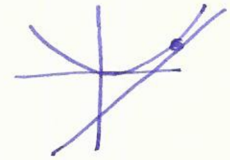
$$= \frac{V^2 (r-R)}{(r+R)^3} = 0 \text{ when } r=R \ldots$$

§3.4 Rates of change

recall: average rate of change $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_1)-f(x_0)}{x_1 - x_0}$



(instantaneous) rate of change $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

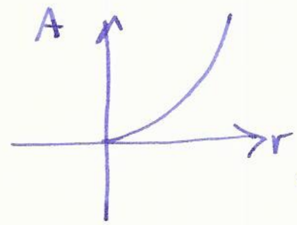


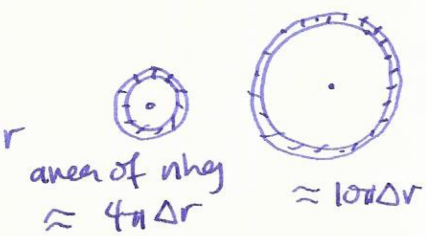
observation: if Δx is small, we can use average rate of change to approximate actual rate of change, and vice versa.

Example: area of circle is $A = \pi r^2$

calculate rate of change of area wrt radius: $\dfrac{dA}{dr} = 2\pi r$

e.g. $\left.\dfrac{dA}{dr}\right|_{r=2} = 4\pi \qquad \left.\dfrac{dA}{dr}\right|_{r=5} = 10\pi$





For small h, $f'(x_0) \approx \dfrac{f(x_0+h) - f(x_0)}{h}$

$\underline{\text{or}}$ $\quad f(x_0+h) \approx f(x_0) + h f'(x_0)$

Example stopping distance in feet given by $F(s) = 1.1s + 0.05s^2$

(s speed in mph)

calculate stopping distance when $s=30$: $F(30) = 1.1 \cdot 30 + 0.05(30)^2$
$= 78$

calculate rate of change of stopping distance wrt speed when $s = 30$

$F'(s) = 1.1 + 0.1s \qquad F'(30) = 1.1 + 0.1 \cdot 30 = 4.1$ ft/mph

estimate stopping distance at $s = 31$ (using above info):

$F(s+h) \approx F(s) + hF'(s)$

$F(31) \approx F(30) + 1 \cdot F'(30) = 78 + 1 \cdot 4.1 = 82.1$