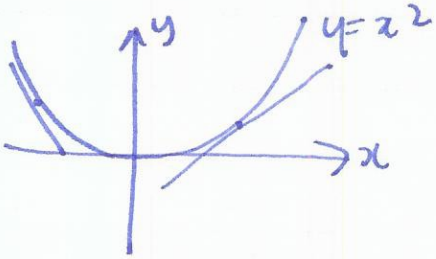
§3.2 Derivative as a function

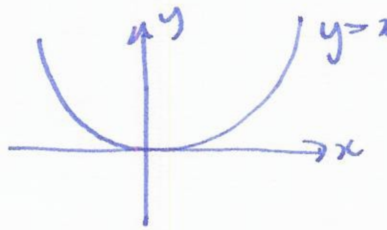


$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

at each input point x, there is a tangent line with a slope, which is a number.
Therefore we can define a function
$x \mapsto$ slope of tangent line at x

<u>notation</u>: we call this function $f'(x)$ or "the derivative of f"

<u>Example</u>

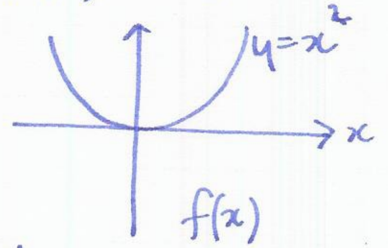


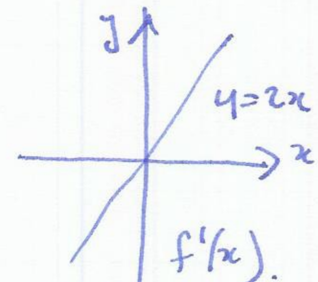
slope at x:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

if $f(x)=x^2$ then $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h}$

$= \lim_{h \to 0} \frac{2xh+h^2}{h} = \lim_{h \to 0} 2x+h = 2x$

<u>summary</u> if $f(x) = x^2$, then $f'(x) = 2x$

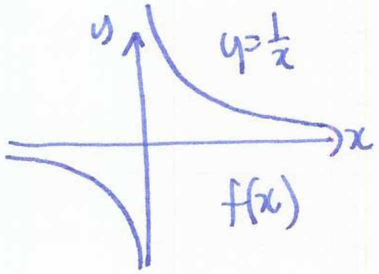


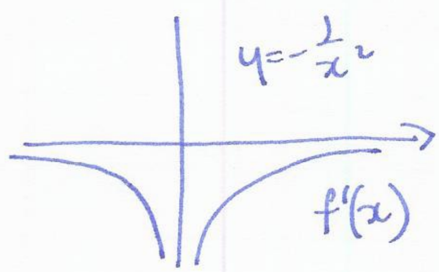


<u>Example</u> $f(x) = \frac{1}{x}$

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{x-(x+h)}{(x+h)x}$

$= \lim_{h \to 0} \frac{-h}{(x+h)hx} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \lim_{h \to 0} \frac{-1}{x^2}$. so $f'(x) = -\frac{1}{x^2}$.





Remarks

① functions : $f: \text{domain} \longrightarrow \text{range}$

e.g. $f: \mathbb{R} \to \mathbb{R}$

~~deri~~ derivative : differentiable functions $\longrightarrow$ functions

$f(x) \longmapsto f'(x)$ or $\frac{df}{dx}$

Warning: not all functions differentiable!

② "calculus" means rules for doing calculations — we don't have to compute explicit limits all the time.

Example $f(x) = x^3$ $\quad f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2.$

Thm (powers of x) if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ (works for all n!)

Example $\frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(x^3) = 3x^2 \quad \frac{d}{dx}(x^4) = 4x^3$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} \quad \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Proof let $f(x) = x^n$, then $f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \dots + h^n}$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $n! = 1.2.3. \dots n.$ all terms have power of $h \geq 2$!

so $\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots}{h} = \lim_{h \to 0} nx^{n-1} + h(\text{stuff})$

$= nx^{n-1}$ □