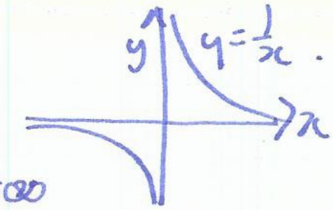
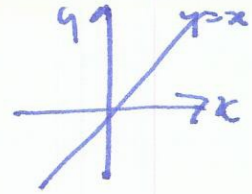
§2.7 Limits at infinity

key observation: $\lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to\infty} x = \infty$





Examples

① $\lim_{x\to\infty} \frac{3x}{2x-1} = \lim_{x\to\infty} \frac{3}{2-1/x} = \frac{3}{2}$

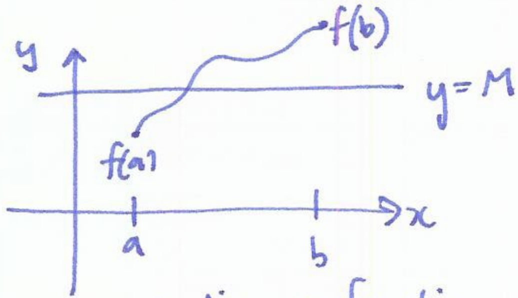
② $\lim_{x\to\infty} \frac{x^2+x}{x-3} = \lim_{x\to\infty} \frac{x+1}{1-3/x} = \infty.$

③ $\lim_{x\to\infty} \frac{1}{x} - \frac{2}{3x+1} = \lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} \frac{2}{3x+1} = 0 - 0 = 0$

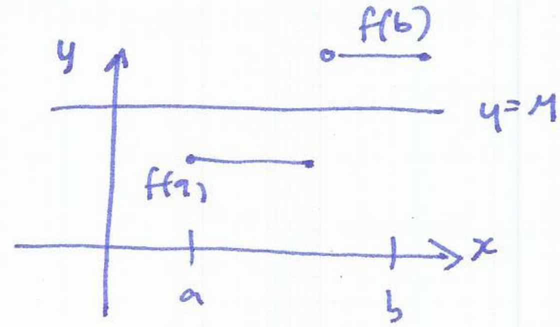
④ $\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{4x+1} = \frac{\sqrt{2+1/x^2}}{4+1} = \frac{\sqrt{2}}{4}.$

§2.8 Intermediate Value Theorem (IVT)

"continuous functions can't skip values"



continuous function

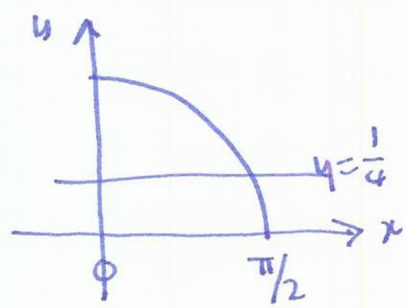


not continuous.

<u>Thm</u> (Intermediate Value Theorem IVT)

If $f(x)$ is a continuous function on a closed interval $[a,b]$ and $f(a) \neq f(b)$, then for every number M between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ such that $f(c) = M$. □

<u>Example</u> show $\cos(x) = \frac{1}{4}$ has at least one solution



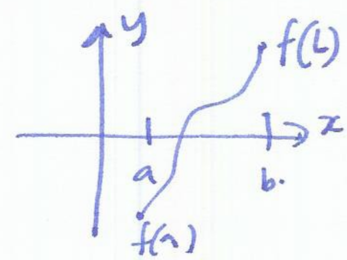
consider $[0, \frac{\pi}{2}]$ $\quad \cos(0) = 1$

$\cos(\frac{\pi}{2}) = 0$

$0 \leq \frac{1}{4} \leq 1 \Rightarrow$ there is $c \in [0, \frac{\pi}{2}]$ with $\cos(c) = \frac{1}{4}$.

<u>Special case</u>: finding zeros.

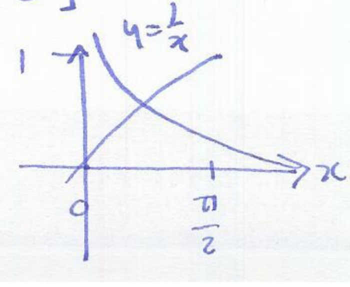
<u>Corollary</u> If $f(x)$ is continuous on $[a,b]$ and $f(a), f(b)$ have different signs, then there is at least one $c \in [a,b]$ with $f(c) = 0$.



<u>Bisection method</u>:

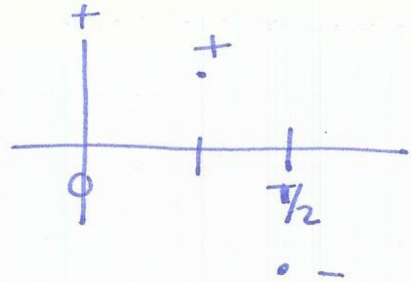
Find a solution to $\sin x = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin(x)$



$f(0) = +\infty$

$f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 \approx -0.36\cdots$



try midpoint: $\frac{\pi}{4}$

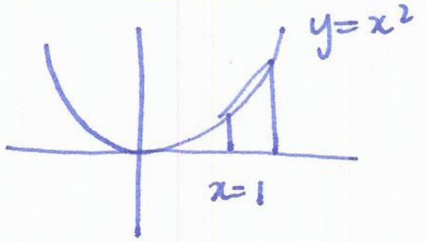
$f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.566\cdots$

so now choose $[\frac{\pi}{4}, \frac{\pi}{2}]$ and choose midpoint $\frac{3\pi}{4}$, and so on...

$f(\frac{\pi}{4}) > 0 \quad f(\frac{\pi}{2}) < 0$

§3.1 Definition of the derivative

<u>recall</u>:



<u>recall</u>: we can compute the average rate of change of a function over some interval $[x_0, x_1]$: $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$

<u>Q</u>: how do we compute the slope of the tangent line?

<u>idea</u>: look at average rate of change over small interval $[x, x+h]$ and take limit as $h \to 0$

<u>Defn</u> the slope of the tangent line at $x = a$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

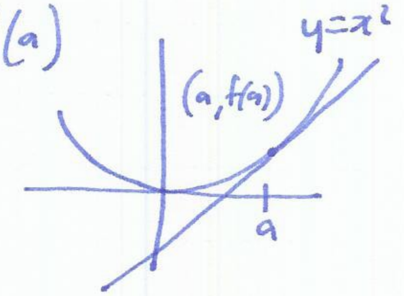
<u>Notation</u>: also called the <u>derivative</u>, written $f'(a)$ or $\frac{df}{dx}(a)$
(Newton) (Leibniz).

if this limit exists we say the function $f(x)$ is <u>differentiable</u> at $x = a$.

<u>note</u>: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ same as $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

<u>Def</u>

Def$^{\underline{n}}$ The tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$



the equation for this line is: $y - y_0 = m(x - x_0)$

i.e. $\quad y - f(a) = f'(a)(x-a)$

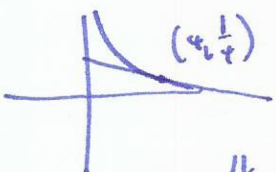
Example find tangent line to $y = x^2$ at $x = 1$

$(x, f(x))$ is $(1,1)$. slope $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0} \dfrac{(1+h)^2 - 1}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = \lim\limits_{h\to 0} 2 + h = 2.$

so equation of tangent line is $y - 1 = 2(x-1) \Leftrightarrow y = 2x - 1$.

Example find slope of tangent to line to $f(x) = \frac{1}{x}$ at $x = 4$.



slope $f'(4) = \lim\limits_{h\to 0} \dfrac{f(4+h) - f(4)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1}{4+h} - \frac{1}{4}}{h}$

$= \lim\limits_{h\to 0} \dfrac{1}{h} \dfrac{4 - (4+h)}{(4+h)4} = \lim\limits_{h\to 0} \dfrac{4-4-h}{4h(4+h)} = \lim\limits_{h\to 0} \dfrac{-h}{4h(4+h)} = \lim\limits_{h\to 0} \dfrac{-1}{4(4+h)} = \dfrac{-1}{16}$

Example straight line $y = mx + b$

slope at $x = a$ $\quad f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h \to 0} \dfrac{m(a+h) + b - ma - b}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{ma} + mh + \cancel{b} - \cancel{ma} - \cancel{b}}{h} = \lim\limits_{h\to 0} \dfrac{mh}{h} = \lim\limits_{h\to 0} m = m.$

Observation if $f(x) = b$ (constant) then $f'(a) = 0$ for all a.