Examples $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2+x^4+1}}$ continuous at $x=1$.

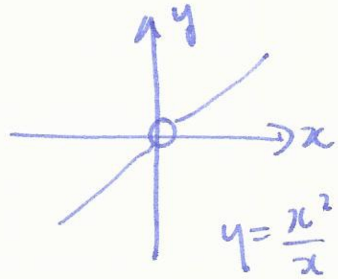
Q: when is $f(x) = \dfrac{x^2}{\sin(x)}$ cts ?

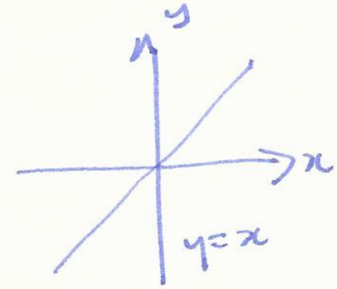
§2.5 Evaluating limits algebraically

Example $\lim\limits_{\cancel{x\to 0}} \dfrac{x^2}{x}$ undefined at $x=0$: $\dfrac{0}{0}$ indeterminate form

but $\lim\limits_{x\to 0} \dfrac{x^2}{x}$ does not depend on value at $x=0$.

$\dfrac{x^2}{x} = x$ for $x \neq 0$





so $\lim\limits_{x\to 0} \dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$.

Indeterminate forms : $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$

note : $\dfrac{1}{0}$ not indeterminate, gives limit $\pm\infty$ or DNE.

Example $\lim\limits_{x\to 3} \dfrac{x^2-4x+3}{x^2+x-12}$ $x=3$: $\dfrac{9-12+3}{9+3-12} = \dfrac{0}{0}$

factor : $\dfrac{(x-3)(x-1)}{(x-3)(x+4)} = \dfrac{x-1}{x+4}$ $(x \neq 3)$

so $\lim\limits_{x\to 3} \dfrac{x^2-4x+3}{x^2+x-12} = \lim\limits_{x\to 3} \dfrac{x-1}{x+4} = \dfrac{2}{7}$

Example $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x}$

$\tan(\pi/2) = \dfrac{\sin(\pi/2)}{\cos(\pi/2)} = 1/0$ undefined

$\sec(\pi/2) = \dfrac{1}{\cos(\pi/2)} = 1/0$ undefined.

but $\dfrac{\tan x}{\sec x} = \dfrac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \sin x$ (if $\cos x \neq 0$)

so $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1$

Example $\displaystyle\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $\qquad \dfrac{2-2}{4-2} = \dfrac{0}{0}$

$\dfrac{\sqrt{x}-2}{x-4} = \dfrac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \dfrac{1}{\sqrt{x}+2}$ & so $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \frac{1}{4}$

Example $\displaystyle\lim_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1}$ $\qquad \dfrac{1}{0} - \dfrac{1}{0}$ undefined.

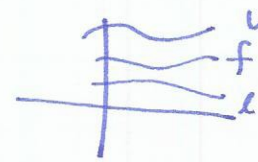
$\dfrac{1}{x-1} - \dfrac{2}{x^2-1} = \dfrac{x+1-2}{x^2-1} = \dfrac{x-1}{x^2-1} = \dfrac{1}{x+1}$ $\qquad \displaystyle\lim_{x\to 1}\frac{1}{x+1} = \frac{1}{2}$

§2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \frac{\sin(x)}{x}$ (substitute $x=0$: $\frac{0}{0}$ undefined)

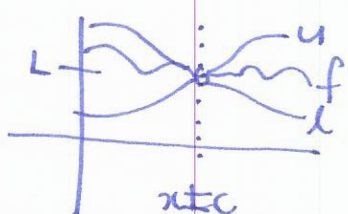
A: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (try $x = 0.1, 0.01, 0.001, \ldots$)

Squeeze theorem: suppose $l(x) \leq f(x) \leq u(x)$



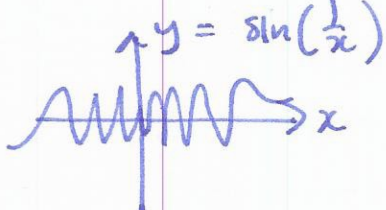
Thm suppose $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$

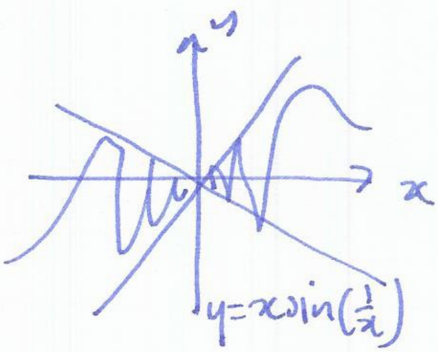
then $\lim_{x\to c} f(x) = L$



$\square$.

Example $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$



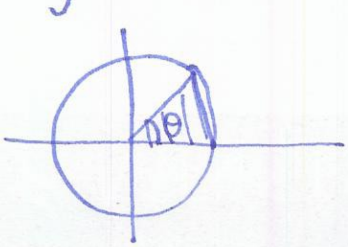


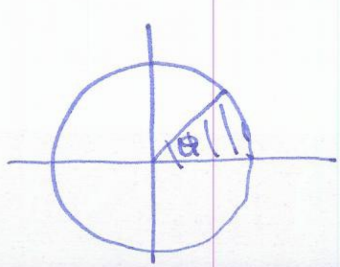
so $-|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$

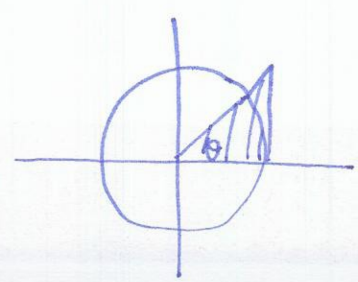
$\lim_{x\to 0} |x| = 0$ $\quad \lim_{x\to 0} -|x| = 0$ $\quad$ so $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

Thm $\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\quad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$

Proof (of $\lim_{x\to 0} \frac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$, consider the following 3 areas







area of triangle $\leq$ area of sector $\leq$ area of triangle

$= \frac{1}{2} bh \qquad = \pi r^2 \frac{\theta}{2\pi} \qquad = \frac{1}{2} bh$

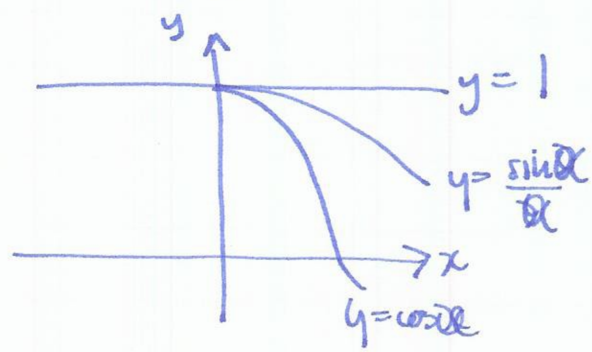
$$= \frac{1}{2} \cdot 1 \cdot \sin\theta \leq \; = \frac{\theta}{2} \; \leq \frac{1}{2} \frac{\sin\theta}{\cos\theta}$$

$$\sin\theta \leq \theta \qquad\qquad \theta \leq \frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \leq 1 \qquad\qquad \cos\theta \leq \frac{\sin\theta}{\theta}.$$

$\Rightarrow$

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$



$$\lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} 1 = 1$$

so squeeze theorem $\Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\square$.

Examples

① $\lim_{x \to 0} \frac{\sin 3x}{2x}$ know $\lim_{x \to 0} \frac{\sin x}{x} = 1$

write this as $\theta = 3x \Leftrightarrow x = \frac{\theta}{3}$.

$$\lim_{\theta \to 0} \frac{\sin\theta}{2\theta/3} = \lim_{\theta \to 0} \frac{3}{2} \frac{\sin\theta}{\theta} = \frac{3}{2} \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \frac{3}{2}.$$

② $$\lim_{t \to 0} \frac{1 - \cos t}{\sin t} = \lim_{t \to 0} \frac{1 - \cos t}{t} \cdot \frac{t}{\sin t} \quad (*)$$

$$\lim_{t \to 0} \frac{1 - \cos t}{t} = 0 \qquad \lim_{t \to 0} \frac{t}{\sin t} = \frac{1}{\lim_{t \to 0} \frac{\sin t}{t}} = 1 \qquad \text{so } (*) = 0 \cdot 1 = 0.$$