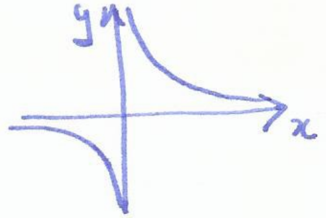
Infinite limits

we say $\lim_{x\to c} f(x) = +\infty$ if $f(x)$ becomes arbitrarily large and positive as $x \to c$

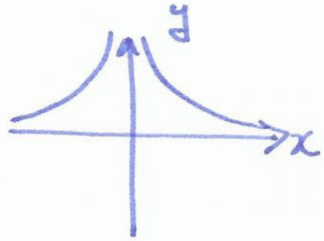
we say $\lim_{x\to c} f(x) = -\infty$ if $f(x)$ .. .. negative as $x \to c$

Example $f(x) = \frac{1}{x}$



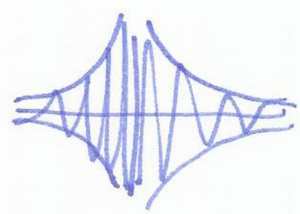
$\lim_{x\to 0+} \frac{1}{x} = +\infty$, $\lim_{x\to 0-} \frac{1}{x} = -\infty$ } different! so $\lim_{x\to 0} \frac{1}{x}$ DNE

Example $f(x) = \frac{1}{x^2}$



$\lim_{x\to 0+} \frac{1}{x^2} = +\infty$, $\lim_{x\to 0-} \frac{1}{x^2} = +\infty$ } same, so $\lim_{x\to 0} \frac{1}{x^2} = +\infty$

Bad example $f(x) = \frac{1}{x}\sin\left(\frac{1}{x}\right)$



no limit as $x \to 0$.

§2.3 Basic limit laws

Example $\lim_{x\to 0} 2x+2 = \lim_{x\to 0} 2x + \lim_{x\to 0} 2 = 0 + 2 = 2$

Thm assume that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist, and are finite.

Then

1) sums: $\lim_{x\to c} \left(f(x) + g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

2) constant multiple: $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ k constant (does not depend on x)

3) products : $\lim_{x\to c} \left(f(x)\, g(x)\right) = \lim_{x\to c} \left(f(x)\right) \lim_{x\to c} \left(g(x)\right)$

4) quotients : $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x\to c} f(x)\right)}{\left(\lim_{x\to c} g(x)\right)}$ as long as $\lim_{x\to c} g(x) \neq 0$.

Warning : these rules don't work if either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ DNE.

Examples :

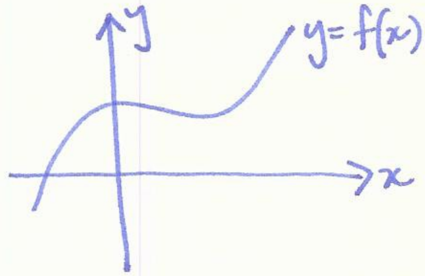
$$\lim_{x\to 3} x^2 = \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) = 3.3 = 9.$$

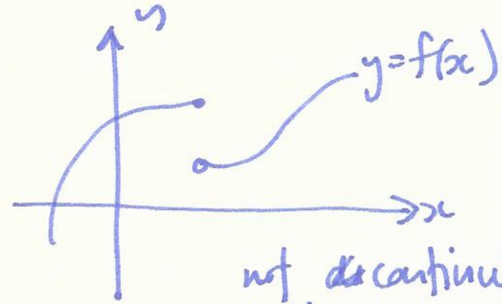
$$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\left(\lim_{t\to 2} t+5\right)}{\left(\lim_{t\to 2} 3t\right)} = \frac{7}{6}$$

Bad examples :

§2.4 Limits and continuity

Example



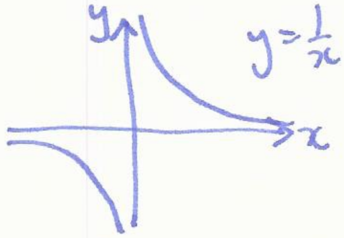


not discontinuous
(discontinuous)

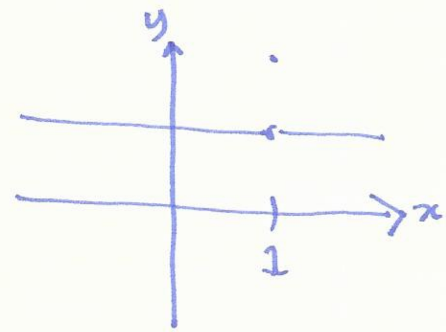
Defn We say $f(x)$ is <u>continuous</u> at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

If the limit does not exist, or is not equal to $f(c)$ then $f(x)$ is not continuous at $x=c$.

Example



$f(x)=\frac{1}{x}$ not continuous at $x=0$

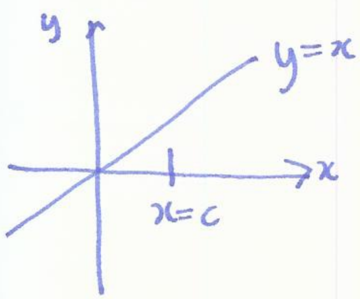


$f(x) = \begin{cases} 1 & x \neq 2 \\ 2 & x = 2 \end{cases}$

not continuous at $x=2$

Example

show $f(x)=x$ is continuous



$f(c) = c$

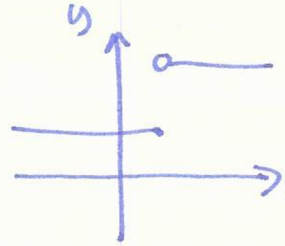
want to show $\lim_{x\to c} f(x) = c$

follows from basic limit law: $\lim_{x\to c} x = c$.

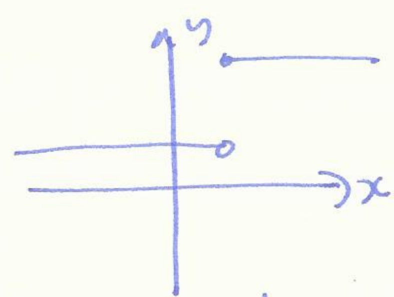
Defn $f(x)$ is <u>left continuous</u> at $x=c$ if $\lim_{x\to c^-} f(x) = f(c)$

<u>right continuous</u> at $x=c$ if $\lim_{x\to c^+} f(x) = f(c)$

Examples



left continuous



right continuous

If at least one of the left or right limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x=c$.

Building continuous functions

Thm 0 $f(x)=k$, $f(x)=x$ are continuous. $\square$

Thm 1 Suppose that $f(x)$ and $g(x)$ are both continuous at $x=c$, then the following functions are continuous at $x=c$

1) $f(x)+g(x)$
2) $kf(x)$ for any constant k
3) $f(x)g(x)$
4) $\dfrac{f(x)}{g(x)}$ if $g(c)\neq 0$

Proof: these follow directly from the limit laws.

check 1) $f(x)$ continuous means $\lim\limits_{x\to c} f(x) = f(c)$

$g(x)$ continuous means $\lim\limits_{x\to c} g(x) = g(c)$

so $\lim\limits_{x\to c}(f(x)+g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x) = f(c)+g(c)$

as required. $\square$.

Thm 2 Polynomials $p(x)$ are continuous.

Rational functions $\frac{p(x)}{q(x)}$ are continuous, except where $q(x)=0$.

Proof $f(x)=x$ is continuous.

so $f(x)f(x) = xx = x^2$ is continuous, similarly x^n. (products)

so $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is cts. (multiply by const and addition)

so $\frac{p(x)}{q(x)}$ is continuous (division) when $q(x) \neq 0$. $\square$

Useful facts

Thm 3
- $\sin(x)$, $\cos(x)$ are continuous
- b^x is continuous $(b>0)$
- $\log_b(x)$ is continuous
- $x^{1/n}$ is continuous

(sometimes called elementary functions)

Thm 4 (inverse functions) If $f: D \to \mathbb{R}$ is continuous, with inverse $f^{-1}: R \to D$ then f^{-1} is continuous.

Thm 5 (composition) If $f(x)$ is continuous at $x=c$ and $g(x)$ is continuous at $x=f(c)$ then $g(f(x))$ is continuous at $x=c$