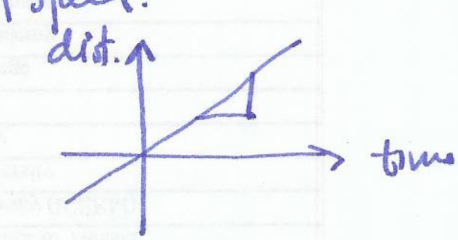
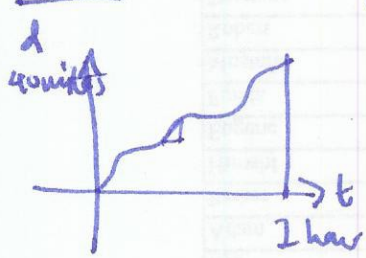
§2.1 Limits, rates of change, tangent lines

motivation: velocity example: driving at constant speed.

velocity $= \dfrac{\text{distance}}{\text{time}} =$ slope of line

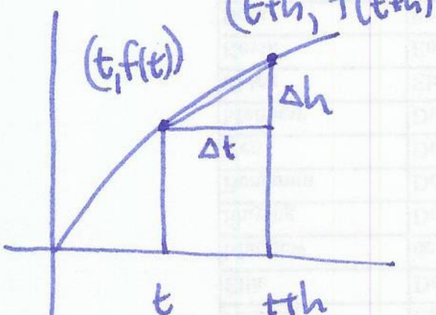


problem: what happens if you don't travel at constant speed?



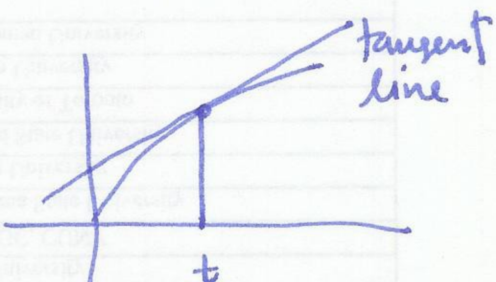
average speed $= \dfrac{\text{distance travelled}}{\text{length of time interval}}$

we can look at average speed over <u>any</u> time interval, including very short ones



average speed on time interval $[t, t+h]$

is $\dfrac{\Delta d}{\Delta t} = \dfrac{f(t+h) - f(t)}{(t+h) - t} = \dfrac{f(t+h) - f(t)}{h}$



Q: what is the speed at time t?
(sometimes called instantaneous speed)

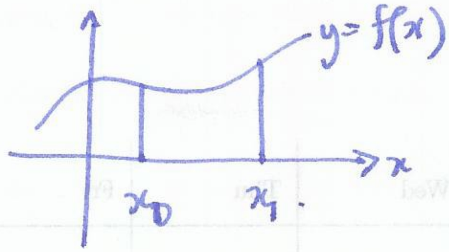
A: speed is slope of tangent line at t.

idea (hope): as length of interval $[t, t+h]$ gets small, the average speed gets close to the slope of the tangent line.
This works for "nice" functions.

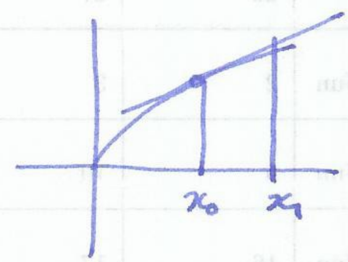
observation: this works for any function $y = f(x)$, not just speed.

<u>summary</u> average rate of change of $f(x)$ over an interval $[x_0, x_1]$

is $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$



§2.2 Limits



<u>aim</u>: want to find slope of tangent lines

<u>know</u>: average rate of change $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

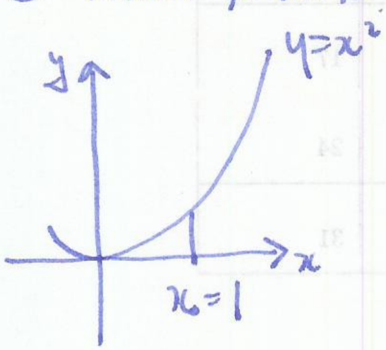
<u>Q</u>: why can't we just set $x_1 = x_0$

<u>A</u>: No. get $\dfrac{f(x_0) - f(x_0)}{x_0 - x_0} = \dfrac{0}{0}$ undefined.

<u>Observations</u>

① if we draw pictures, the average slope gets closer to the slope of the tangent lines as the length of the interval gets small.

② seems to work for sample calculations too:



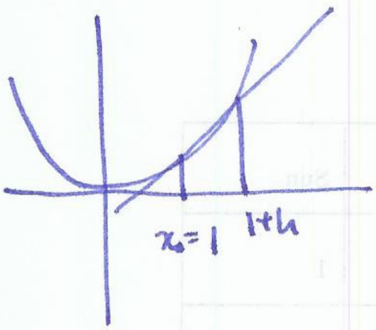
$x_1 = 2$: $\dfrac{f(2) - f(1)}{2 - 1} = \dfrac{4 - 1}{1} = 3$

$x_1 = 1\frac{1}{2}$: $\dfrac{f(3/2) - f(1)}{3/2 - 1} = \dfrac{9/4 - 1}{1/2} = \dfrac{5}{2} = 2.5$

$x_2 = 1.1$: $\dfrac{1.21 - 1}{0.1} = 2.1$

$x_1 = 1.01$: $\dfrac{1.0201 - 1}{0.01} \approx 2.01$

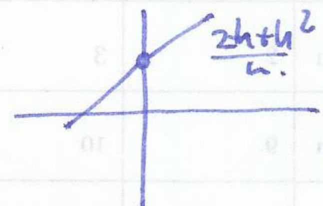
3 seems to work algebraically

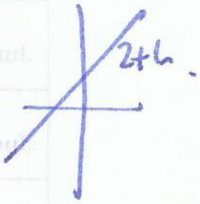


average rate of change from $\underset{x_0}{1}$ to $\underset{x_1}{1+h}$

$$= \frac{f(1+h)-f(1)}{1+h-1} = \frac{(1+h)^2-1^2}{h} = \frac{1+2h+h^2-1}{h}$$

$$= \frac{2h+h^2}{h} \underset{(h\neq 0)}{=} 2+h$$





<u>Defn</u> Let f be a function defined on an interval containing c, but not necessarily defined at c. We say

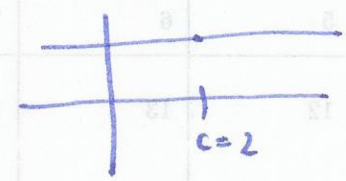
"the limit of $f(x)$ as x approaches c is equal to L"

written as: $\lim_{x\to c} f(x) = L$ or $f(x)\to L$ as $x\to c$

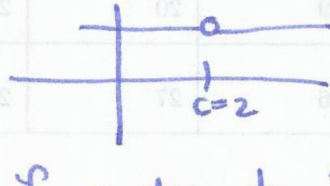
if $|f(x)-L|$ becomes arbitrarily small as x gets close to c

also say: "$f(x)$ converges to L as x tends to c"

<u>Examples</u> a) $f(x)=5$ $c=2$



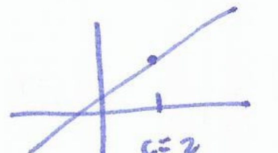
b) $f(x) = \frac{5x}{x}$, $c=2$



want to show: $|f(x)-5|$ close to 0 if x close to 2

as $|f(x)-5| = |5-5| = 0$ for all $x\neq 2$, so this is true.

c) $\lim_{x\to 2} 2x+1 = 5$



want to show $|f(x)-5|$ close to zero when x close to 2

$$|f(x)-5| = |2x+1-5| = |2x-4| = 2|x-2|$$

x close to 2 $\Leftrightarrow$ $|x-2|$ small.

<u>useful facts</u>

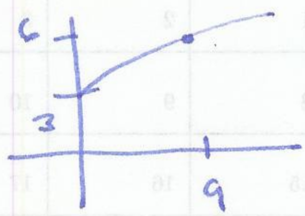
<u>Thm</u> for any constants k, c $\quad \lim_{x\to c} k = k$

$$\lim_{x\to c} x = c$$

Investigating limits: try
- drawing a picture
- calculate close values
- algebra

<u>Example</u> $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$ problem: can't plug in $x=9$, get $\frac{0}{0}$.

- draw picture:



looks like $x=6$

- calculate:

x	$\frac{x-9}{\sqrt{x}-3}$
8.9	5.98329
9.1	6.016
8.99	5.99833
9.01	6.00166

- algebra

difference of two squares

$$x-9 = (\sqrt{x}-3)(\sqrt{x}+3)$$

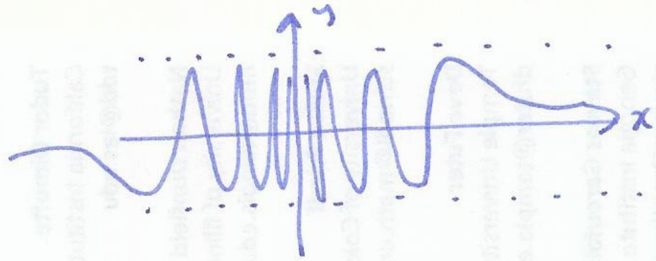
$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3}$$

$$= \sqrt{x}+3 \quad (x \neq 9).$$

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} \sqrt{x}+3 = 6$$

<u>Bad example</u> : no limit

$f(x) = \sin\left(\frac{1}{x}\right)$



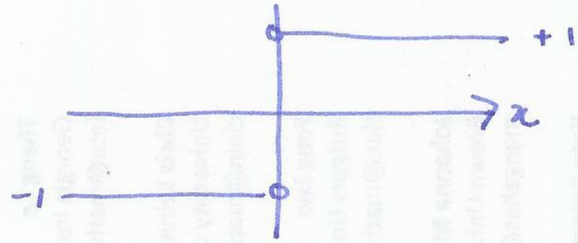
no limit near $x = 0$

<u>note</u> : $f\left(\frac{1}{2\pi n}\right) = \sin(2\pi n) = 0$

$f\left(\frac{1}{2\pi n + \frac{\pi}{2}}\right) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1$

<u>One sided limits</u>

<u>Example</u> $f(x) = \frac{x}{|x|}$



$f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$

sometimes useful to distinguish left from right limits

<u>notation</u> : $\lim_{x \to 0+} f(x)$ means right limit (only consider $x > 0$)

$\lim_{x \to 0-} f(x)$ means left limit (only consider $x < 0$)

<u>note</u> : in order for the two sided limit $\lim_{x \to c} f(x)$ to exist, the right limit must equal the left limit: $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x)$.

<u>Example</u> $f(x) = \frac{x}{|x|}$

$$\left.\begin{aligned} \lim_{x \to 0+} f(x) &= 1 \\ \lim_{x \to 0-} f(x) &= -1 \end{aligned}\right\} \text{ different so } \lim_{x \to 0} f(x) \text{ DNE}$$