Math 231 Calculus I

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office 1S-222   office hours M 12:20 – 2:15
W 12:20 – 1:10

- math tutoring: 1S-214
- students with disabilities

Text: Early transcendentals, by Rogawski (1st or 2nd ed)

§1.2 Linear and quadratic functions

recall:

$$\begin{tikzcd} \text{input set} \arrow[r, mapsto, "\text{function}"] & \text{output set} \\ \text{domain} & \text{range} \end{tikzcd}$$

examples:

$$f : \mathbb{R} \longrightarrow \mathbb{R} \qquad (\mathbb{R} = \text{real numbers})$$

$$x \longmapsto x^2 \qquad (\text{description of function})$$

e.g.

$$0 \longmapsto 0$$
$$1 \longmapsto 1$$
$$-1 \longmapsto 1$$
$$2 \longmapsto 4 \quad \text{etc.}$$

notation:

$$\underset{\text{name}}{f} : \underset{\substack{\uparrow \\ \text{input/domain}}}{\mathbb{R}} \longrightarrow \mathbb{R} \ \nwarrow \text{output/range}$$

examples:

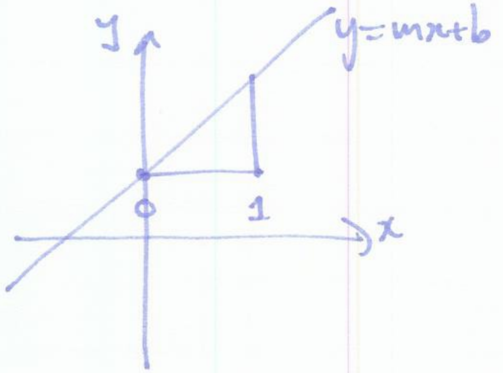
$$+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(a, b) \longmapsto a + b$$

evaluation at 0 : $\left\{ \begin{matrix} \text{functions} \\ f: \mathbb{R} \to \mathbb{R} \end{matrix} \right\} \longrightarrow \mathbb{R}$

$$f \longmapsto f(0)$$

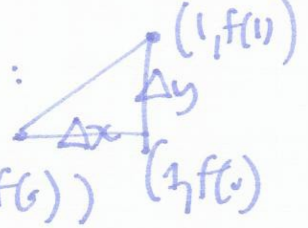
<u>key property</u>: every input goes to a <u>unique</u> output (not a collection of outputs!)

A linear function $f: \mathbb{R} \to \mathbb{R}$ has the form $f(x) = mx + b$ (m, b constants, i.e. don't depend on x). The graph of a linear function is a straight line.



slope $= \dfrac{\text{vertical change}}{\text{horizontal change}} = \dfrac{\Delta y}{\Delta x}$

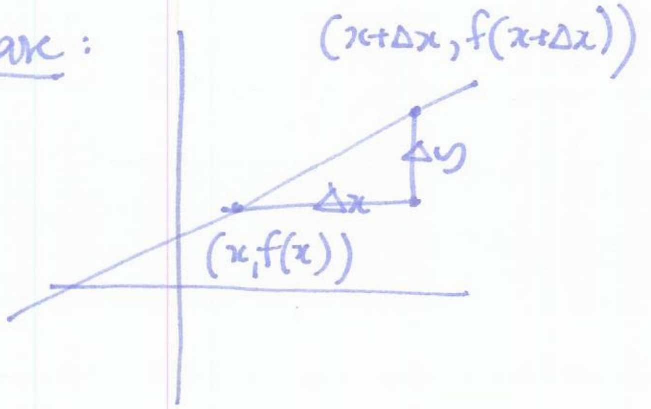
<u>special case</u>:



$$\cancel{\frac{\Delta x}{\Delta y}} \frac{\Delta y}{\Delta x} = \frac{f(1) - f(0)}{1}$$

$$= \frac{m + b - b}{1} = m$$

<u>general case</u>:

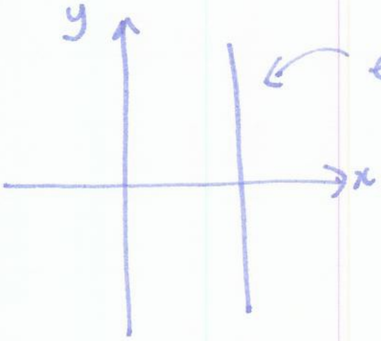


$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{x + \Delta x - x} = \frac{m(x + \Delta x) + b - (mx + b)}{\Delta x}$$

$$= \frac{\cancel{mx} + m\Delta x + \cancel{b} - \cancel{mx} - \cancel{b}}{\Delta x} = \frac{m\Delta x}{\Delta x} = m$$

<u>useful fact</u>: a straight line has constant slope m everywhere.

<u>observations</u>:
- $|m|$ large, line is steep
- $m=0$ horizontal line
- $m>0$ goes up from left to right
- $m<0$ " down " " "
- vertical lines are not graphs of functions.



equation of vertical line is $x=c$

equation vs function

equation: relation between variables e.g
$x=c$
$x^2+y^2=1$

function: a map from one set to another
e.g. $f:\mathbb{R}\to\mathbb{R}$.

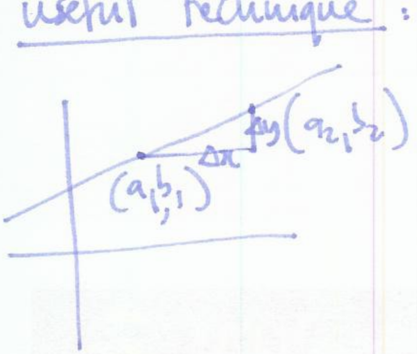
the graph of $f:\mathbb{R}\to\mathbb{R}$ is $y=f(x)$ (an equation!)
but not every equation comes from a function.

To deal with any straight line, use the general linear equation

$ax+by=c$ (at least one of a,b not zero).

e.g $y=mx+b$ set $b=1$ above $x=c$ set $a=0$.

useful technique: find equation of line through two points

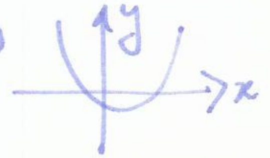


- find slope: $m=\dfrac{\Delta y}{\Delta x}=\dfrac{b_2-b_1}{a_2-a_1}$
- line: $y-b_1=m(x-a_1)$

Quadratic functions

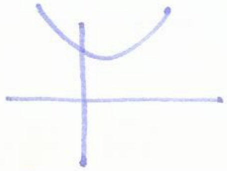
given by $f(x) = ax^2+bx+c$ (a, b, c constants, i.e. do not depend on x)

graphs as parabolas

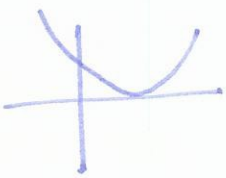


at most two distinct real solutions to $f(x)=0$

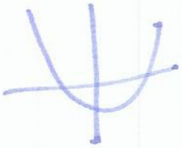
given by $f(x) = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$



$b^2-4ac<0$
no real sol^ns



$b^2-4ac=0$
one distinct real solution



$b^2-4ac>0$
two distinct real sol^ns.

useful techniques

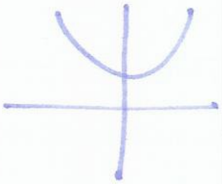
- factorization : $ax^2+bx+c = a(x-r_1)(x-r_2)$ r_1, r_2 solutions or roots.
- complete the square : any quadratic function can be written as $(x+a)^2+b$

example :

x^2+2x+3

$(x+1)^2+2$

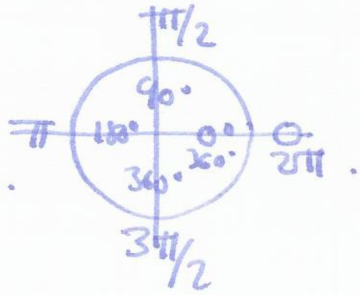
$x^2+2x+1+2$ no sol^ns!



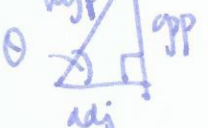
§1.4 Trig functions

- angles vs radians radians win.

angle in radians = dist travelled around unit circle.



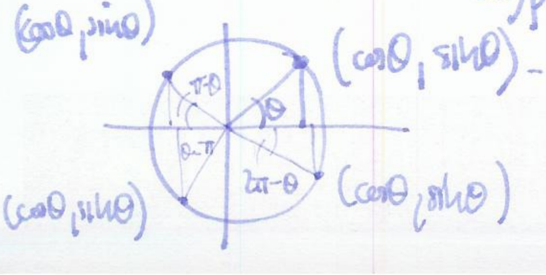
- right angled triangles



$\sin\theta = \dfrac{opp}{hyp}$

$\cos\theta = \dfrac{adj}{hyp}$

} can extend these functions to be defined for all $\theta \in \mathbb{R}$.



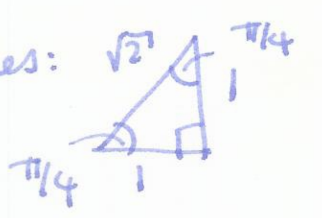
useful facts: $\sin(-\theta) = -\sin(\theta)$ (odd)

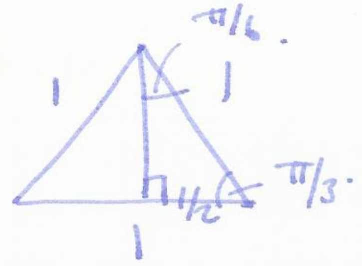
$\cos(-\theta) = \cos\theta$ (even).

- special values:

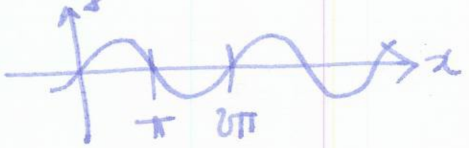
θ	0	$\pi/2$	$\pi/4$	$\pi/3$	$\pi/6$.
$\sin\theta$	0	1	$\sqrt{2}/2$	$\sqrt{3}/2$	$1/2$
$\cos\theta$	1	0	$\sqrt{2}/2$	$1/2$	$\sqrt{3}/2$

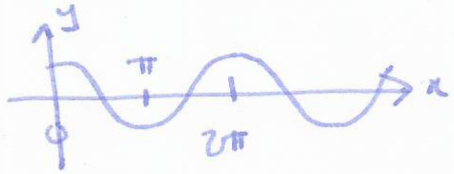
from special triangles:





- the graphs of $\sin\theta, \cos\theta$ are periodic with period 2π



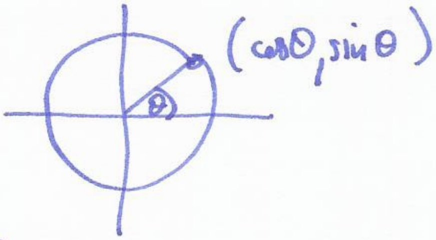


<u>other trig functions</u> $\tan(x) = \dfrac{\sin(x)}{\cos(x)} = \dfrac{\text{opp}}{\text{adj}}$

$\text{cosec}(x) = \dfrac{1}{\sin x}$ $\sec(x) = \dfrac{1}{\cos(x)}$ $\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$

<u>Trig identities</u>

<u>Pythagorean identity</u>: $\cos^2 x + \sin^2 x = 1$



3 for the price of one!

$$\frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \quad \leftrightarrow \quad \cot^2 x + 1 = \text{cosec}^2 x$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \leftrightarrow \quad 1 + \tan^2 x = \sec^2 x$$

<u>Double angle</u>:

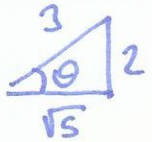
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

Addition $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \sin x - \sin x \sin y$

special case (shift): $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$

Example suppose $\sin\theta = \frac{2}{3}$ find $\cos\theta$, $\tan\theta$, $\sin 2\theta$

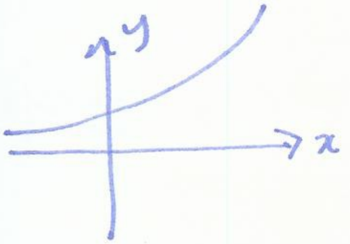


$\cos\theta = \frac{\sqrt{5}}{3}$ $\tan\theta = \frac{2}{\sqrt{5}}$ $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \cdot \frac{2}{3} \cdot \frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$

§1.6 Exponential and logarithmic functions

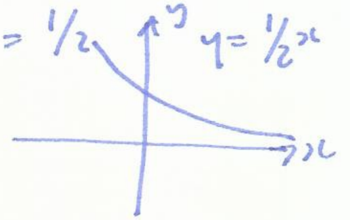
Example $x \mapsto 2^x$



x	-2	1	0	1	2	3
$f(x)$	$1/4$	$1/2$	1	2	4	8

can use any <u>positive</u> number instead of 2

$f(x) = b^x$ $(b>0)$ e.g $b = 1/2$



useful properties:

- positive $b^x > 0$ for all x
- b^x increasing if $b > 1$
  decreasing if $0 < b < 1$
- b^x grows faster than any polynomial x^n

exponent rules:

$b^0 = 1$ $b^x b^y = b^{x+y}$, $b^{-x} = \frac{1}{b^x}$, $\frac{b^x}{b^y} = b^{x-y}$

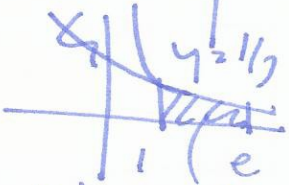
$(b^x)^y = b^{xy}$, $b^{1/n} = \sqrt[n]{b}$

- there is a special exponential function e^x, $e = 2.71828\ldots$

key properties:

① e is the unique number such that e^x has slope 1 at $x=0$

② e is the unique number such that the area under $y=1/x$ between 1 and e has area 1.



Logarithms

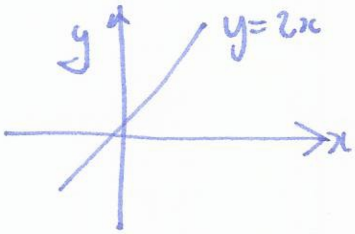
The logarithm is the inverse function for the exponential function.

Recall

$$\begin{tikzcd} \text{domain} \arrow[r, bend left, "f"] & \text{range} \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Warning $f^{-1}(x) \neq \dfrac{1}{f(x)}.$

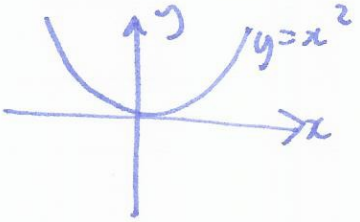
Example $f(x)=2x$



$f^{-1}(x)=\frac{1}{2}x$

$$\begin{tikzcd} 3 \arrow[r, mapsto, bend left, "x\mapsto 2x"] & 6 \arrow[l, bend left, "x\mapsto \frac{1}{2}x"] \end{tikzcd}$$

Bad example $f(x)=x^2$



$1 \longmapsto 1$

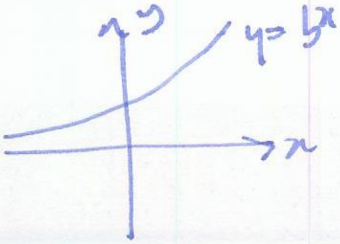
$-1 \longmapsto 1$

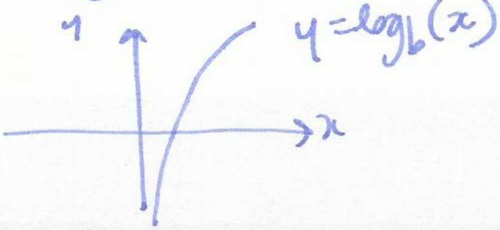
f^{-1}? no inverse!

A function has an inverse if it passes the horizontal line test

i.e. every horizontal line hits the graph of the function exactly once.

The exponential function $y=b^x$ has an inverse called the logarithm (to base b) written $f^{-1}(x)=\log_b(x)$





useful fact: graph of $f^{-1}(x)$ is graph of $f(x)$ reflected in $y=x$

special logarithm with $b=e$ is called the natural logarithm

notation: $\log_e(x) = \ln(x)$

- inverse function properties: $f(f^{-1}(x)) = x$ $\quad f^{-1}(f(x)) = x$

so $b^{\log_b(x)} = x \qquad \log_b(b^x) = x$

logarithm rules:

$$b^0 = 1 \leftrightarrow \log_b(1) = 0$$

$$b^1 = b \leftrightarrow \log_b(b) = 1$$

$$b^x b^y = b^{x+y} \leftrightarrow \log_b(st) = \log_b(s) + \log_b(t)$$

$$\frac{1}{b^x} = b^{-x} \leftrightarrow \log_b\left(\frac{1}{t}\right) = -\log_b(t)$$

$$\frac{b^x}{b^y} = \cancel{\log}\, b^{x-y} \leftrightarrow \log\left(\frac{s}{t}\right) = \log(s) - \log(t)$$

$$(b^x)^y = b^{xy} \leftrightarrow \log_b(s^t) = t\log_b(s)$$

converting between bases:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} \quad \text{for any } a.$$

so $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$