Finding potential functions:

example (2d): $\underline{F}(x,y) = \langle y, x \rangle = \nabla f$ for some f.

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = y \Rightarrow f(x,y) = xy + c_1(y) \\ \dfrac{\partial f}{\partial y} = x \Rightarrow f(x,y) = xy + c_2(x) \end{array}\right\} \text{ but } \frac{\partial}{\partial y} c_1 = 0, \quad \frac{\partial}{\partial x} c_2 = 0$$

$\Rightarrow f(x,y) = xy + c$, c constant.

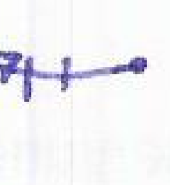
Divergence 2d: $\underline{F}(x,y) = \langle A, B \rangle$ $\quad \nabla \cdot \underline{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}$ (scalar!).

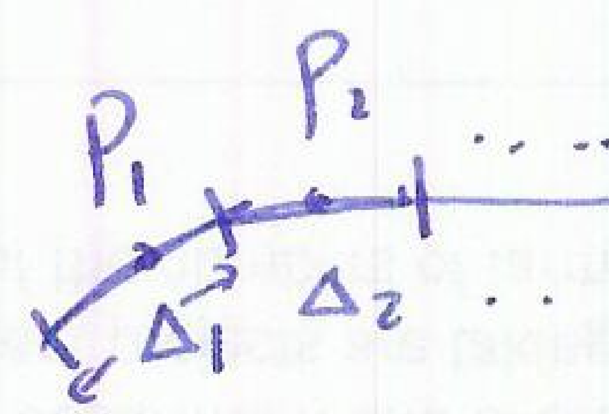
3d: $\underline{F}(x,y,z) = \langle A, B, C \rangle$ $\quad \nabla \cdot \underline{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$ (scalar!).

(means $\nabla \cdot \underline{F} = 0$: fluid flow incompressible).

useful fact: $\nabla \cdot \nabla \times \underline{F} = 0$

§16.2 Line integrals

(parameterized) curve C 



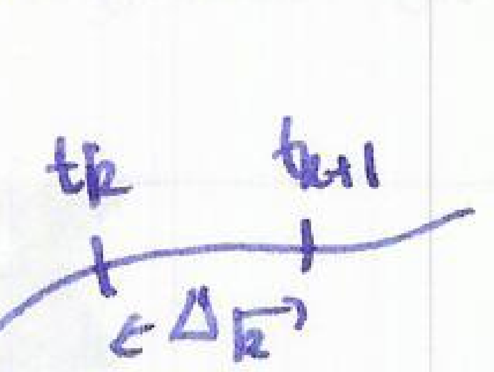
scalar line integral $\int_C f(x,y,z)\, ds$ $\quad$ ($f(x,y,z)$ scalar function $f: \mathbb{R}^3 \to \mathbb{R}$)

can define as Riemann sum $\sum f(x_i) ds$ $\quad \int_C f(x,y,z)\, ds = \lim_{|\Delta s_i| \to 0} \sum_{i=1}^{n} f(P_i) \Delta_i$

Thm $\underline{c}(t)$ $a \le t \le b$ parameterized curve C

then $\int_C f(x,y,z)\, ds = \int_a^b f(\underline{c}(t)) \, \|\underline{c}'(t)\| \, dt$ $\leftarrow$ does not depend on choice of parameterization

Note if $f(x,y,z) = 1$ then get $\int_a^b \|c'(t)\| \, dt$ = arc length

Intuition: 

length of this segment is $\int_{t_k}^{t_{k+1}} \|c'(t)\| \, dt$

so want $\lim\limits_{n\to\infty} \sum_{k=1}^{n} f(t_k) \int_{t_k}^{t_{k+1}} \|c'(t)\|\,dt \sim \int_a^b f(t)\,\|c'(t)\|\,dt$.

Notation $ds = \|c'(t)\|\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$.

Example find $\int_C (x+y+z)\,ds$ where C is the helix $\underline{c}(t) = (\cos t, \sin t, t)$

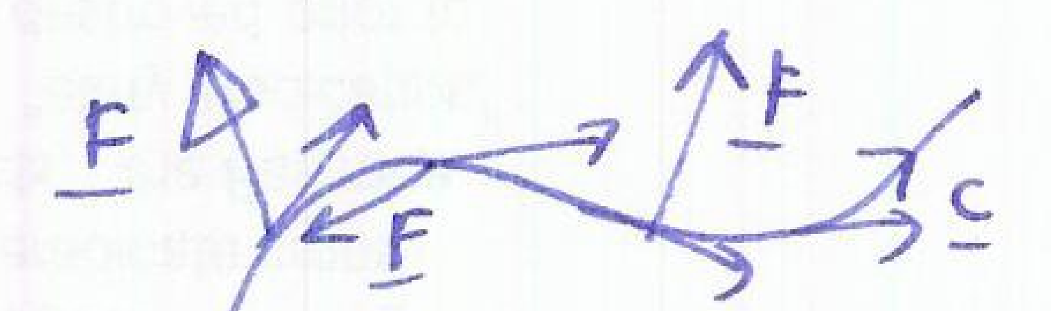
$\underline{c}'(t) = (-\sin t, \cos t, 1)$.

for $0 \le t \le 2\pi$

$= \int_0^{2\pi} (\cos t + \sin t + t)(\sin^2 t + \cos^2 t + 1)^{1/2}\,dt = \int_0^{2\pi} 2(\cos t + \sin t + t)\,dt$.

$= 2\left[+\sin t - \cos t + \frac{1}{2}t^2\right]_0^{2\pi} = \frac{1}{2}(2\pi)^2 = 2\pi^2$

Vector line integrals $\int_C \underline{F}\cdot d\underline{s}$.



intuition: integral of tangential component of $\underline{F}$ along C

unit tangent vector! $\rightarrow \dfrac{c'(t)}{\|c'(t)\|}$

i.e. $\sum \underline{F}(x,y,z)\cdot \underline{c}'(t)\,\Delta t_i$

Def^{n} Let T = unit tangent vectors to $\underline{c}$. (i.e. $\underline{T} = \dfrac{c'(t)}{\|\underline{c}'(t)\|}$).

$\int_C \underline{F}\cdot d\underline{s} = \int_C (\underline{F}\cdot\underline{T})\,ds$.

If C has parameterization $c(t)$:

$\int_C \underline{F}\cdot d\underline{s} = \int_a^b \underline{F}(\underline{c}(t))\cdot \dfrac{\underline{c}'(t)}{\|\underline{c}(t)\|}\,\|\underline{c}'(t)\|\,dt$.

$$\boxed{\int_C \underline{F}\cdot d\underline{s} = \int_a^b \underline{F}(\underline{c}(t))\cdot \underline{c}'(t)\,dt}$$

Alternate notation: $\int_C \underline{F}\cdot d\underline{s}$ $\qquad \underline{F} = \langle F_1, F_2, F_3 \rangle$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

to evaluate this with parameterization $\underline{c}(t) = \langle x(t), y(t), z(t) \rangle$

$$\int_C \underline{F}\cdot d\underline{s} = \int_a^b \left(F_1(c(t))\frac{dx}{dt}\cancel{dt} + F_2(\underline{c}(t))\frac{dy}{dt} + F_3(\underline{c}(t))\frac{dz}{dt} \right) dt$$ same as before.

Example (2d) integrate $\underline{F} = \langle 2y, -3 \rangle$ over the ellipse

$\underline{c}(t) = (4 + 3\cos\theta,\ 3 + 2\sin\theta) \quad 0 \le \theta \le 2\pi.$

$\underline{c}'(t) = (-3\sin\theta,\ 2\cos\theta)$

$$\int_0^{2\pi} \underline{F}\cdot ds = \int_0^{2\pi} \langle \cancel{4\cos\theta}\ 6+4\sin\theta, -3 \rangle \cdot \langle -3\sin\theta, 2\cos\theta \rangle\, d\theta$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \cancel{\int_0^{2\pi} -12\cos\theta\sin\theta - 6\cos\theta\, d\theta = \int_0^{2\pi} -6\sin 2\theta - 6\cos\theta\, d\theta}$$

$\cos 2\theta = 1 - 2\sin^2\theta.$

$$= \cancel{\left[+3\cos 2\theta - 6\sin\theta \right]_0^{2\pi}}$$

$$= \int_0^{2\pi} -18\sin\theta - 12\sin^2\theta - 6\cos\theta\, d\theta = \int_0^{2\pi} -12\sin^2\theta\, d\theta.$$

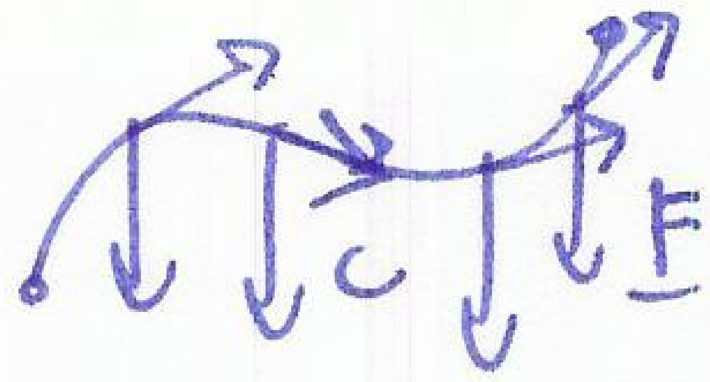
$$= \cancel{k}\int_0^{2\pi} -6 + 6\cos 2\theta\, d\theta = -6\int_0^{2\pi} d\theta = -12\pi.$$

useful properties: $\int_C (\underline{F} + \underline{G})\,ds = \int_C \underline{F}\cdot ds + \int_C \underline{G}\cdot d\underline{s}$

$$\int_C k\underline{F}\cdot ds = k\int_C \underline{F}\,ds$$

reverse orientation: $\int_C \underline{F}\cdot d\underline{s} = -\int_{-C} \underline{F}\cdot d\underline{s}$

Physical interpretation



work done $W = \int_C \underline{F} \cdot d\underline{s}$.

§16.3 Conservative vector fields

In general $\int_C \underline{F} \cdot d\underline{s}$ depends on the path C, not just the endpoints

For special $\underline{F}$, $\int_C \underline{F} \cdot d\underline{s}$ only depends on the endpoints

these vector fields are called conservative, i.e.

if $\underline{C_1}$ and $\underline{C_2}$ are two paths from a to b then

$\underline{F}$ conservative $\Rightarrow \int_{C_1} \underline{F} \cdot d\underline{s} = \int_{C_2} \underline{F} \cdot d\underline{s}$.

Special case : C is a closed curve : then $\underline{F}$ conservative $\Rightarrow \int_C \underline{F} \cdot d\underline{s} = 0$

(C closed, integral sometimes written $\oint_C \underline{F} \cdot d\underline{s}$.)

recall if $\underline{F}$ is a gradient vector ~~conservative then~~ field if $\underline{F} = \nabla f$ for some f.

Thm (fundamental thm for gradient vector fields)

If $\underline{F} = \nabla f$ on domain D, then for every oriented curve C in D with initial point P and final point Q, $\int_C \underline{F} \cdot d\underline{s} = f(Q) - f(P)$.

(If C is closed $\oint_C \underline{F} \cdot d\underline{s} = 0$).

Example $\underline{F} = \langle y+2xz, x, x^2 \rangle$ $\quad \int_C \underline{F} \cdot ds$ $\quad$ C = upper half unit circle oriented clockwise in xy-plane

Q: is $F = \cancel{\alpha}\nabla f$ for some f?

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+2xz & x & x^2 \end{vmatrix} = \langle 0-0, -2x+2x, 1-1 \rangle = \langle 0,0,0 \rangle$$

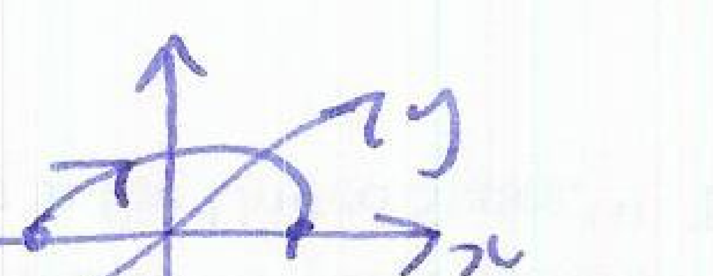
yes.

find f:

$$\left.\begin{aligned} \int y+2xz\,dx &= xy + x^2 z + f_1(y,z) \\ \int x\,dy &= xy + f_2(x,z) \\ \int x^2\,dz &= x^2 z + f_3(x,y) \end{aligned}\right\}$$

simplest function is. $f(x,y,z) = xy + x^2 z$. check!

$$\int_C \underline{F} \cdot ds = f(\underline{b}) - f(\underline{a})$$

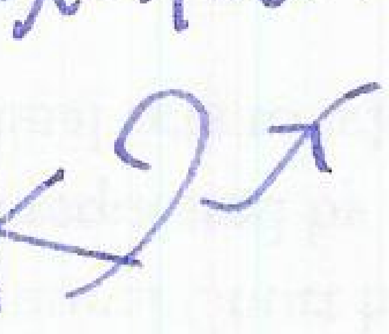


$$= f(1,0,0) - f(-1,0,0) = 0.$$

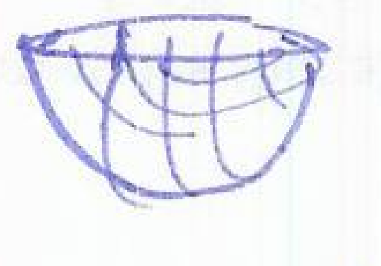
§16.4 Paramaterized surfaces and surface integrals

paramaterized curve: $c(t) : \mathbb{R} \to \mathbb{R}^3$

$t \mapsto (x(t), y(t), z(t))$.

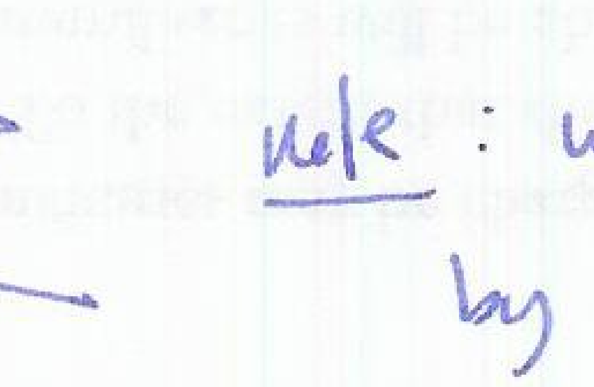


parameterized surface: $\mathbb{R}^2 \to \mathbb{R}^3$



$(u,v) \xrightarrow{\phi} (x(u,v), y(u,v), z(u,v))$

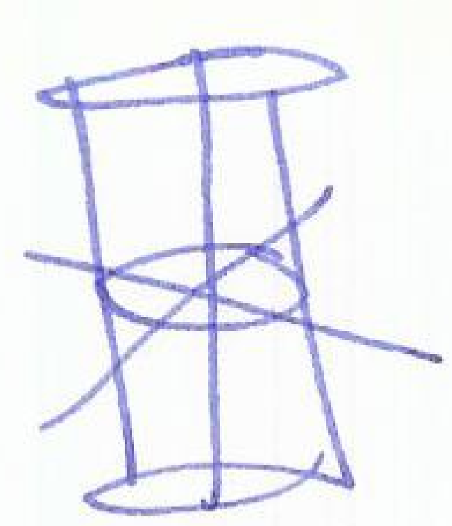
Example ① paraboloid $z = x^2 + y^2$.



note: we can parameterize this by $(u,v) \mapsto (u,v,f(u,v))$!

$\phi(u,v) = (u, v, u^2+v^2)$.

② cylinder



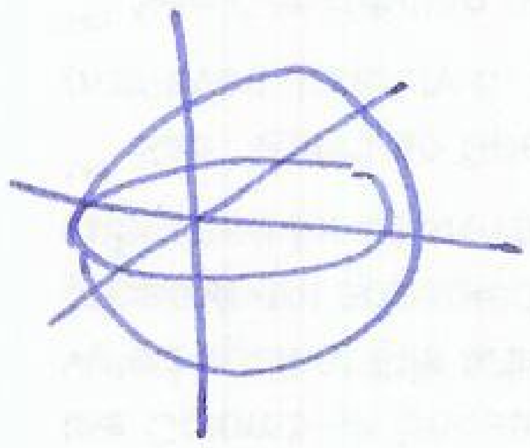
$x^2+y^2=1$.

$(u,v) \mapsto$

$(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$

important: $0 \le \theta \le 2\pi$

③ sphere:



spherical polar coords!

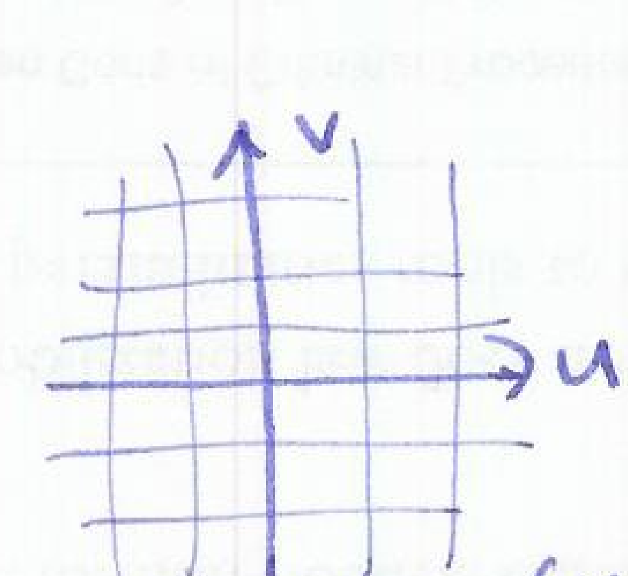
$(\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$.

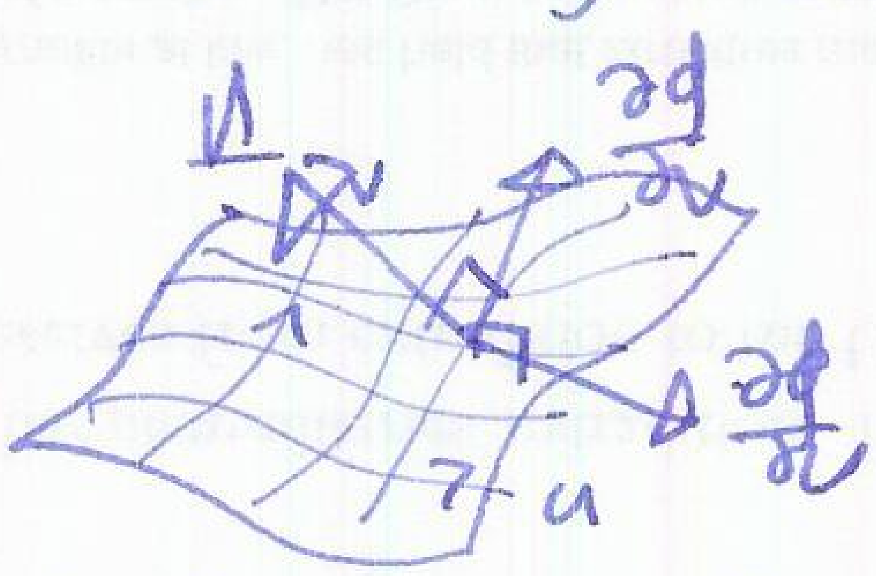
important: $0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

④ any graph $z = f(x,y)$ can be parameterized by $(u,v) \mapsto (u, v, f(u,v))$

Coordinate lines:





$(u,v) \mapsto \phi(u,v) = (x(u,v), y(u,v), z(u,v))$.

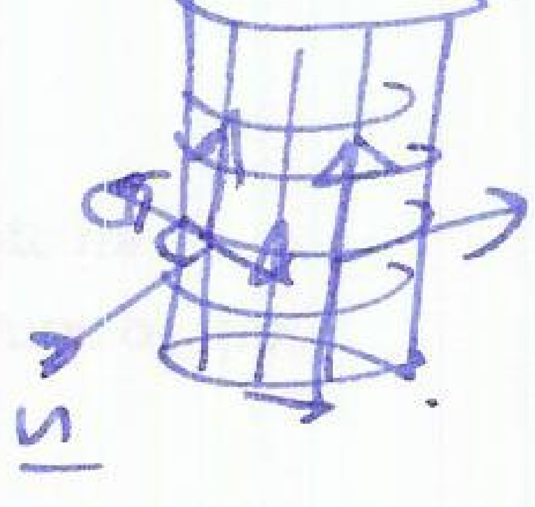
$\frac{\partial \phi}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ = tangent vector to surface in u-direction

$\frac{\partial \phi}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$ = tangent vector to surface in v-direction.

Q: how do we find the normal vector to the surface?

$\underline{n} = \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}$ is normal vector

Example cylinder: $(\theta, z) \overset{\phi}{\mapsto} (\cos\theta, \sin\theta, z)$.

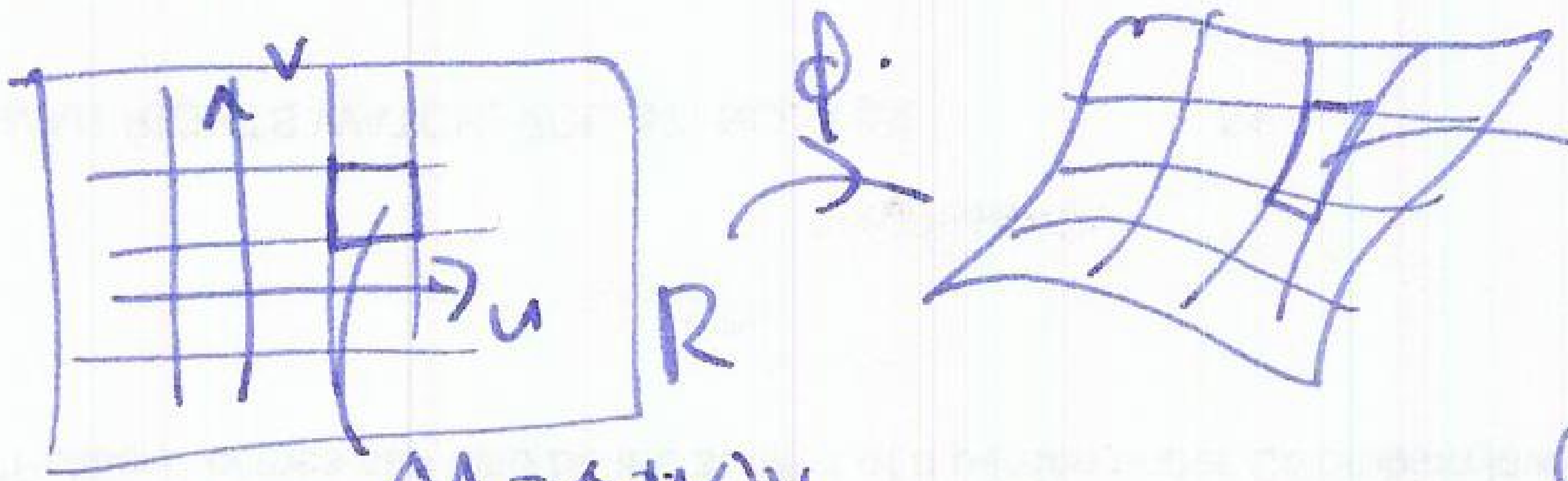


$\frac{\partial \phi}{\partial \theta} = (-\sin\theta, \cos\theta, 0)$

$\frac{\partial \phi}{\partial z} = (0, 0, 1)$

$$\underline{n} = \frac{\partial \phi}{\partial \theta} \times \frac{\partial \phi}{\partial z} = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle.$$

Surface area $\phi : \mathbb{R}^2 \to \mathbb{R}^3$.

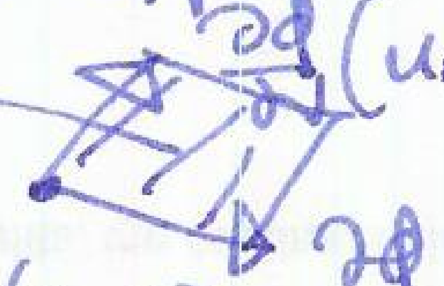


need area scale factor for this piece.

$\Delta A = \Delta u \Delta v$

$dA = du\,dv$

linear approx:



use area of parallelogram

$$= \left\| \frac{\partial\phi}{\partial u} \times \frac{\partial\phi}{\partial v} \right\| = \underline{n}(u,v).$$

so area of surface $= \iint_R \|\underline{n}(u,v)\|\, du\,dv$

area(S)

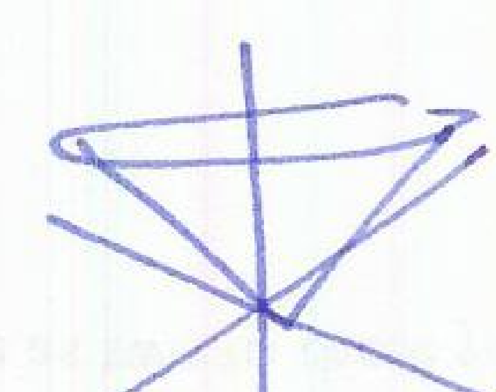
how to integrate a scalar function $f(x,y,z)$ over S:

$$\iint_S f(x,y,z)\, dS = \iint_R f(x,y,z)\, \|\underline{n}(u,v)\|\, du\,dv$$

↑ notation. ↑ with choice of parametrization.

with $dS = \|\underline{n}(u,v)\|\, du\,dv$.

Example find surface area of cone



$\phi(\theta,t) = (t\cos\theta, t\sin\theta, t)$

$0 \le t \le 2$

$0 \le \theta \le 2\pi$

find $\underline{n}$: $\frac{\partial\phi}{\partial\theta} = (-t\sin\theta, t\cos\theta, 0)$

$\frac{\partial\phi}{\partial t} = (\cos\theta, \sin\theta, 1)$

$\underline{n} = \langle t\cos\theta, t\sin\theta, -t \rangle$.

$\|\underline{n}\| = \sqrt{2t^2} = \sqrt{2}|t|$

$$\int_0^{2\pi}\int_0^2 1 \cdot \sqrt{2}\,t \, dt\, d\theta = \left[\frac{\sqrt{2}t^2}{2}\right]_0^2 = 2\sqrt{2}.$$

$$\int_0^{2\pi} 2\sqrt{2}\, d\theta = 4\sqrt{2}\pi.$$

§16.5 Vector integrals over surfaces

integrate a vector field $\underline{F}$ over S:

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_S (\underline{F} \cdot \underline{e_n}) \, dS$$

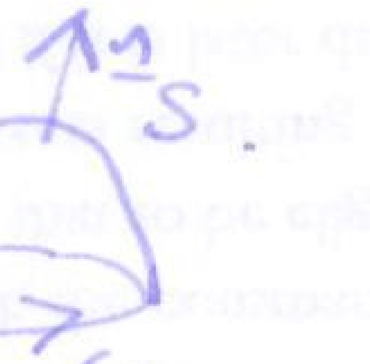
notation

this is also called the flux of $\underline{F}$ across S.

in terms of a parametrization $\Phi(u,v)$ for S: $\displaystyle\iint_D \underline{F}(\Phi(u,v)) \cdot \underline{n}(u,v) \, du\, dv$.

Example $\underline{F}$ = fluid flow

$$\iint_S \underline{F} \cdot d\underline{S} = \text{amount of flow through surface.}$$

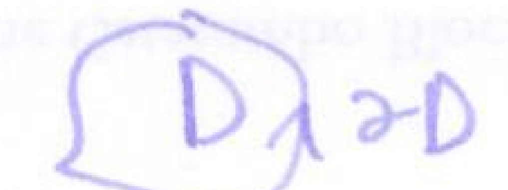


Electricity $\underline{E}$ electric field

$\underline{B}$ magnetic field

Faraday's law of induction: $\displaystyle\int_C \underline{E} \cdot d\underline{s} = -\frac{d}{dt} \iint_S \underline{B} \cdot d\underline{S}$.

§17.1 Green's Theorem

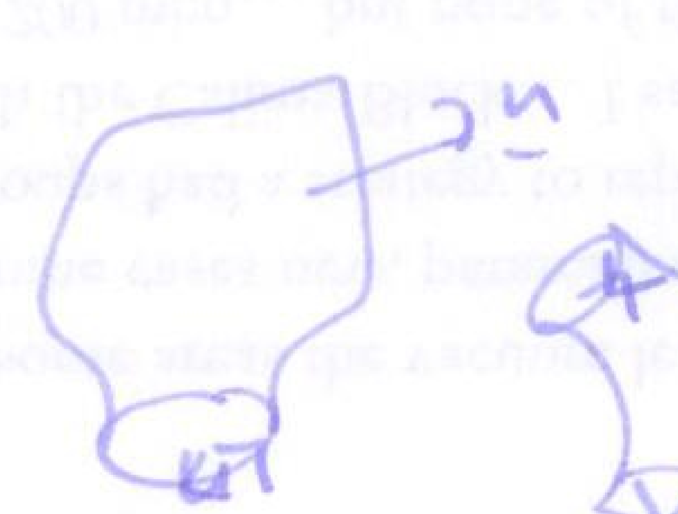


$P(x,y)$ $Q(x,y)$.

$$\underline{\int_{\partial D} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA}$$

$$\int_{\partial D} \underline{F} \cdot d\underline{s} \qquad \int_D \text{"}\nabla \times \underline{F}\text{"} \, dA.$$

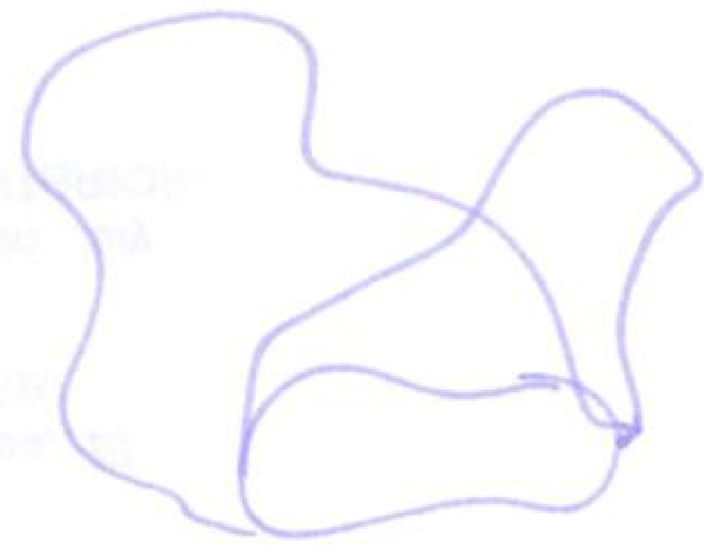
§17.2 Stoke's Theorem



boundary orientation: walk around boundary standing up in normal direction: surface on left.

Thm **Stokes Thm** S surface $\underline{F}$ vector field

$$\int_{\partial S} \underline{F} \cdot \underline{ds} = \iint_S \nabla \times \underline{F} \, d\underline{S}.$$



Thm surface independence for curl vector fields.

i.e. if $\underline{F} = \nabla \times A$ then $\iint_S \underline{F} \, d\underline{S} = \underbrace{\int_{\partial S} \underline{A} \cdot \underline{ds}}_{\text{doesn't depend on } S \text{ only } \partial S!}$

§17.3 Divergence theorem

$W \subset \mathbb{R}^3$ ∂W smooth surface, normal vectors point out.

$\underline{F}$ vector field **Thm** $\iint_{\partial W} \underline{F} \cdot \underline{dS} = \iiint_W \nabla \cdot \underline{F} \, dV.$

$\underline{\nabla \cdot F}$ measures compression/expansion of fluid flow.

Final Wed 16^{th} May 12:20 – 2:15pm 75-219

ww Tue 15^{th}

sample final.