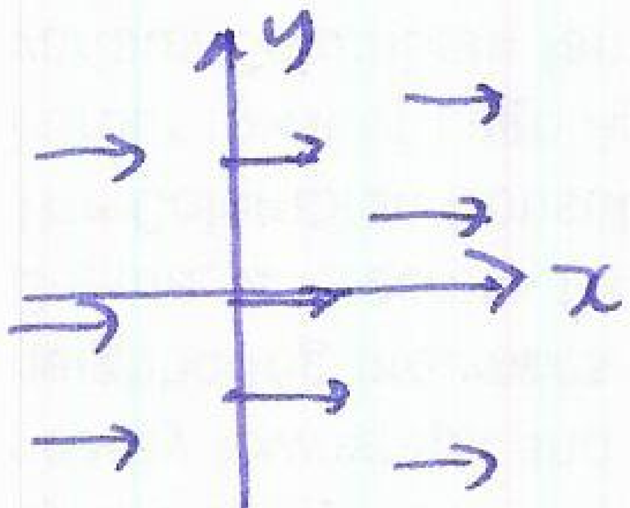
§16.1 Vector fields

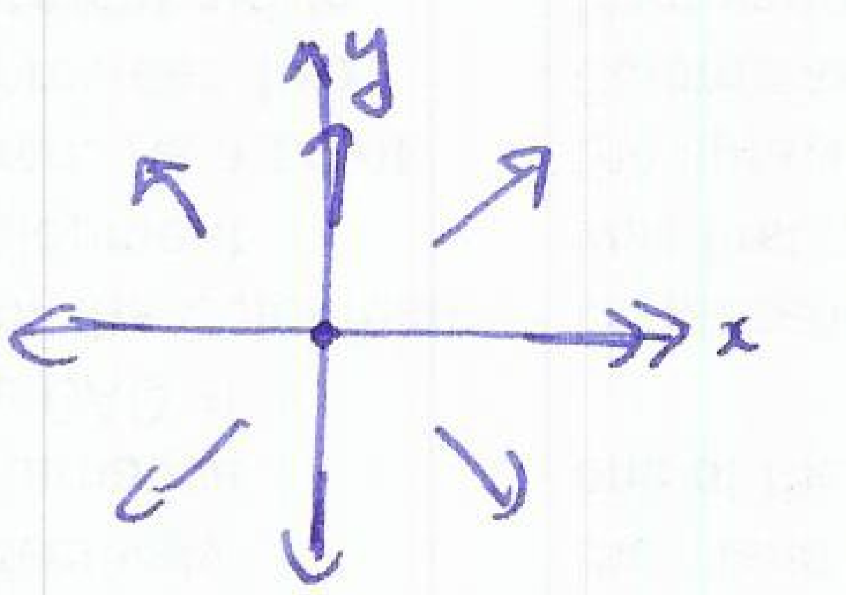
Def A vector field is a function which assigns a vector to each point in the domain.

Examples (2d) ① $\underline{F}(x,y) = \langle 1, 0 \rangle$



$F: \mathbb{R}^2 \to \mathbb{R}^2$.

② $\underline{F}(x,y) = \langle x, y \rangle$



③ any gradient vector field $\underline{F} = \nabla f \qquad f: \mathbb{R}^2 \to \mathbb{R}$.

Def A vector field is conservative if $\underline{F} = \nabla f$ for some scalar function f; called the potential function for $\underline{F}$.

Example (3d) (inverse square field) $\underline{F}(x,y,z) = \underline{F}(\underline{x}) = \dfrac{1}{\|\underline{x}\|^3}\,\underline{x}$.

(note $\|\underline{F}(\underline{x})\| = \dfrac{1}{\|\underline{x}\|^2}$)

claim: this is conservative, with potential function $f(\underline{x}) = -\dfrac{1}{\|\underline{x}\|} = \dfrac{-1}{\sqrt{x^2+y^2+z^2}}$.

check: $\nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right\rangle = \left\langle +\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x, \dots \right\rangle$.

$= \dfrac{\langle x, y, z \rangle}{(\sqrt{x^2+y^2+z^2})^3} = \dfrac{\underline{x}}{\|\underline{x}\|^3}$.

Warning: not all vector fields are conservative.

2d: **Th** $\underline{F}(x,y) = \langle A(x,y), B(x,y) \rangle$ is conservative iff $\dfrac{\partial B}{\partial x} = \dfrac{\partial A}{\partial y}$.

3d: **Def** curl $\underline{F} = \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix} = \langle C_y - B_z, -C_x + A_z, B_x - A_y \rangle$

$\underline{F} = \langle A, B, C \rangle$

Th $\underline{F}$ is conservative iff $\nabla \times \underline{F} = \underline{0}$.