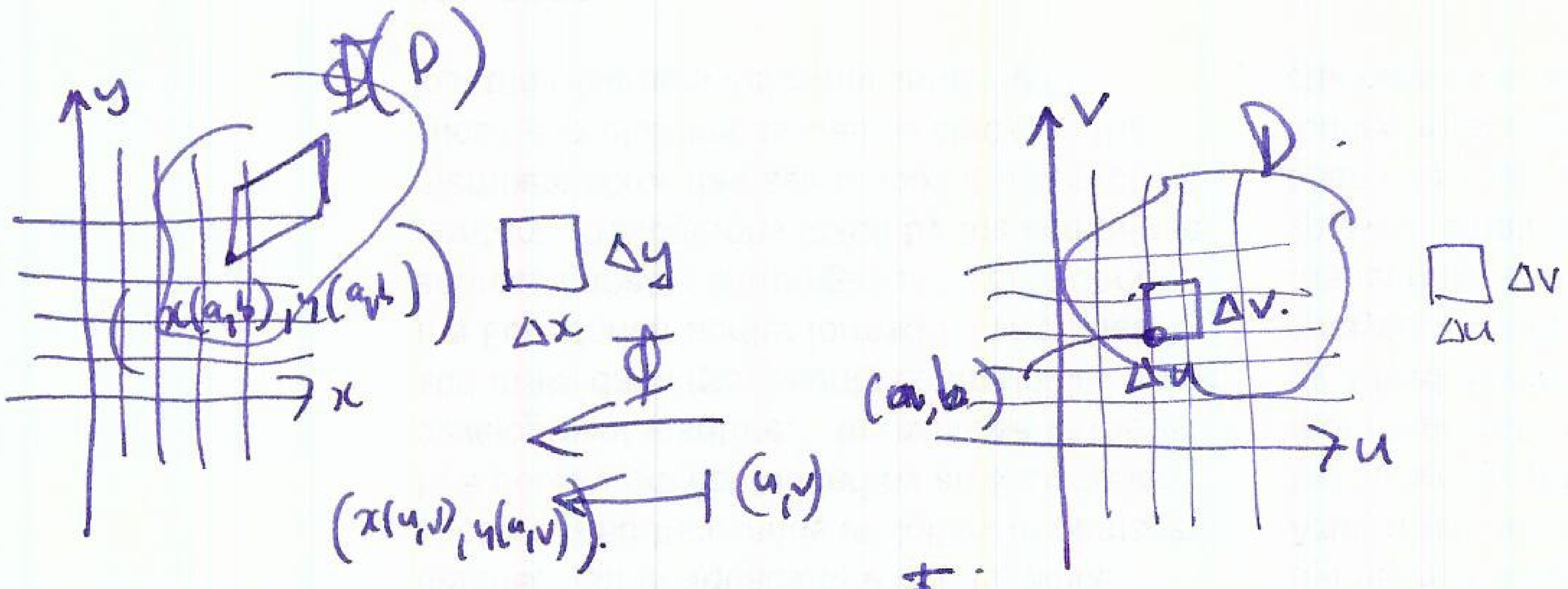
§15.5 Change of variable

1d. $\displaystyle\int_a^b f(x)\,dx \quad \overset{x(u)}{\longleftrightarrow} \quad \int_c^d f(x(u))\frac{dx}{du}\,du$ where $x(d)=b$, $x(c)=a$

2d $\displaystyle\iint_D f(x,y)\,\underset{dxdy}{dA} \quad \overset{x(u,v)}{\underset{y(u,v)}{\longleftrightarrow}} \quad \iint_D f(x(u,v),y(u,v))\,J\,du\,dv.$

in term of x,y in term of u,v.

$$J = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$



$$\Phi: (u,v) \longmapsto (x(u,v), y(u,v))$$

linear approximation at (a,b):

$$(u,v) \longmapsto \Big(x(a,b) + \frac{\partial x}{\partial u}(a,b)(u-a) + \frac{\partial x}{\partial v}(a,b)(v-b),$$
$$y(a,b) + \frac{\partial y}{\partial u}(a,b)(u-a) + \frac{\partial y}{\partial v}(v-b)\Big).$$

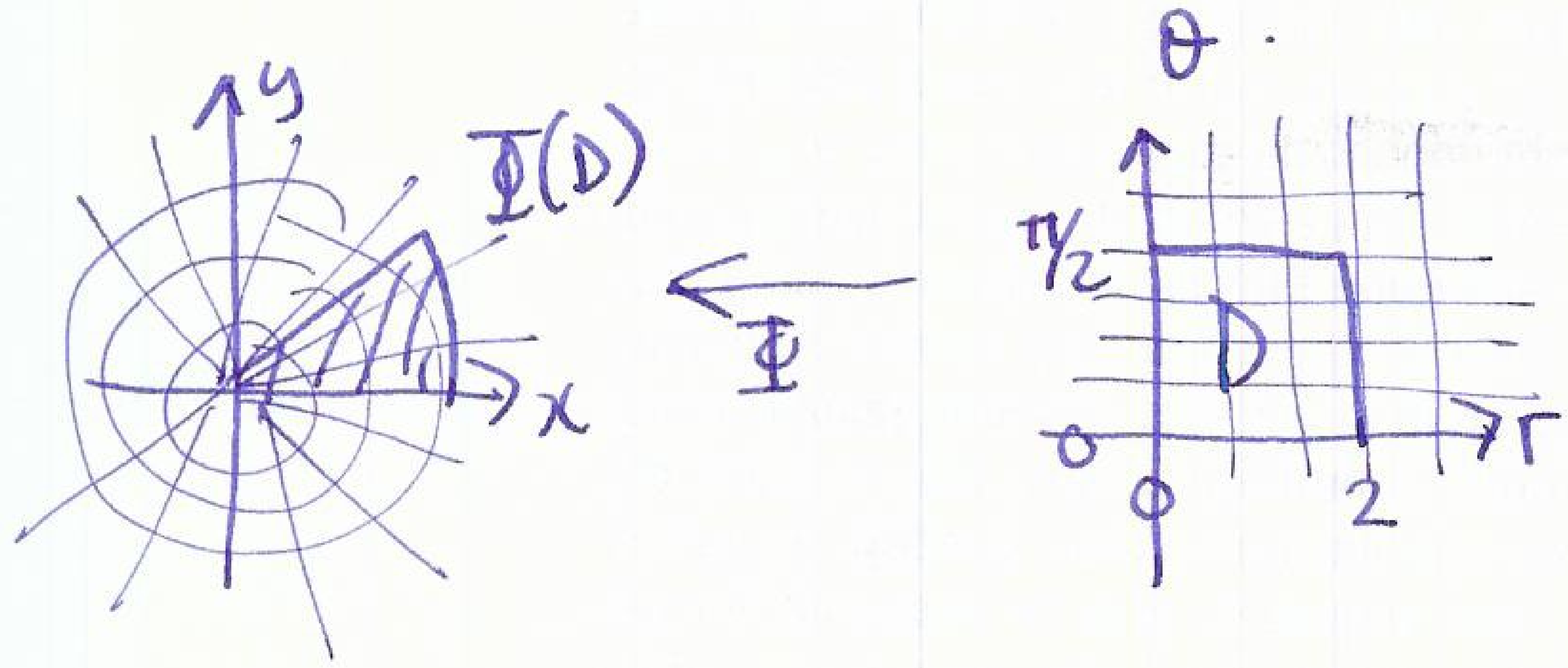
$$= (x(a,b), y(a,b)) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{bmatrix} \begin{bmatrix} u-a \\ v-b \end{bmatrix}.$$

need to replace

$\Delta x \Delta y$ with $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \Delta u \Delta v.$ or $dxdy = J\,du\,dv.$

$J(\Phi)\,du\,dv.$

Example polar coordinates



$$\Phi(r,\theta) = (\overset{x}{r\cos\theta}, \overset{y}{r\sin\theta})$$

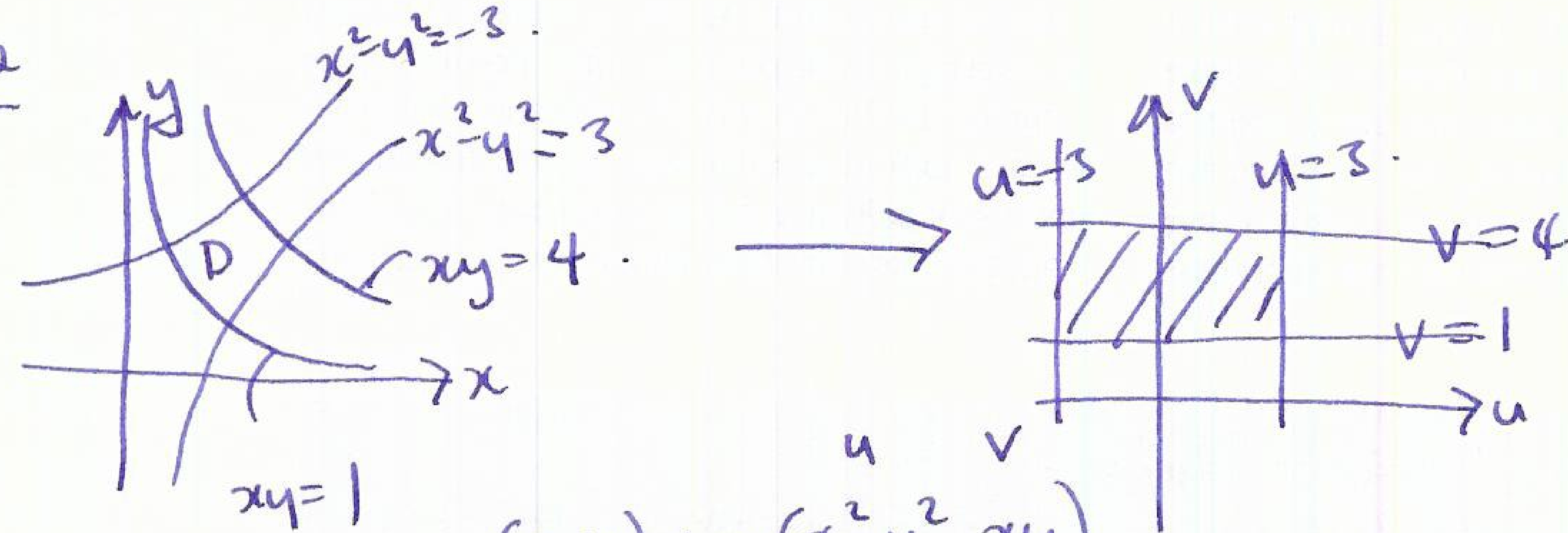
compute $J(\Phi)$.

$$J(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

so $dxdy = r\,dr\,d\theta$.

Useful fact $J(\Phi) = \dfrac{1}{J(\Phi^{-1})}$. $\iff$ $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|}$.

Example



$$(x,y) \mapsto (\overset{u}{x^2-y^2}, \overset{v}{xy})$$

$dxdy = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| dudv$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2 + 2y^2$$

so $dxdy = \dfrac{1}{2(x^2+y^2)}\,dudv$

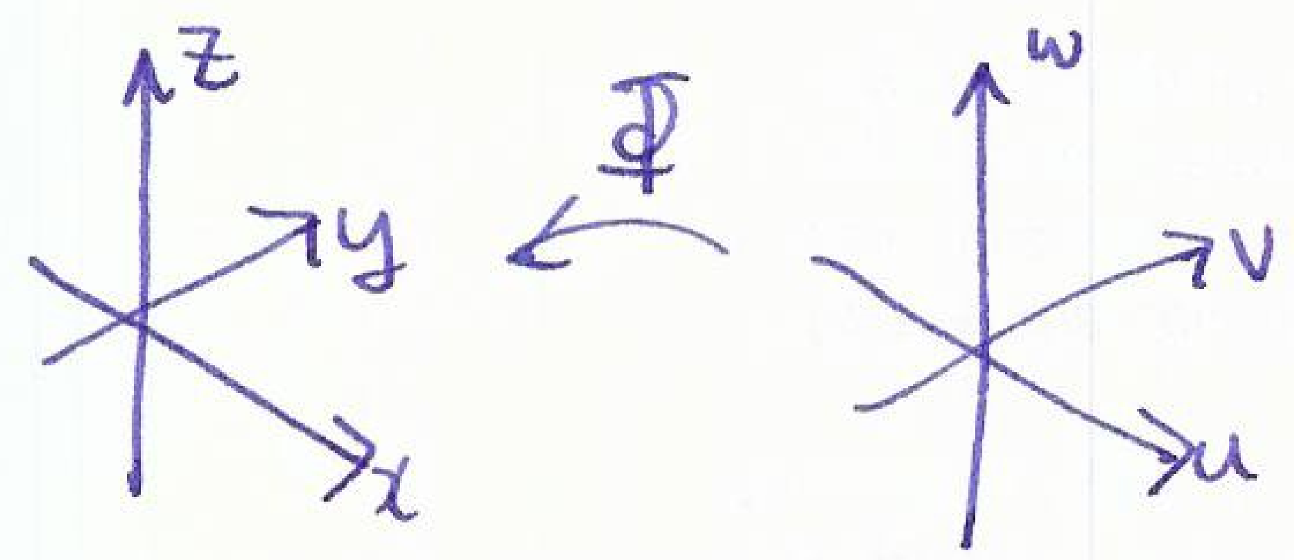
(note: usually need to get this in terms of u,v to do the integral).

find x^2+y^2: $x^2+y^2 = x^2-y^2+2y^2 = u+2y^2$.

find y^2: $v=xy \Rightarrow x = v/y$ so $\dfrac{v^2}{y^2} - y^2 = u \iff y^4 + uy^2 - v^2 = 0 \iff y = \dfrac{-u \pm \sqrt{u^2+4v^2}}{2}$.

so $x^2+y^2 = u+2y^2 = \sqrt{u^2+4v^2}$ ie $dxdy = \dfrac{1}{2\sqrt{u^2+4v^2}}\,dudv$.

3 variables



$$dx\,dy\,dz = |J|\,du\,dv\,dz$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| .$$

Example spherical coordinates.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}.$$

$$= \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\phi \sin\theta & \rho \cos\theta \cos\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}.$$

$$= \cos\phi \begin{vmatrix} -\rho \sin\theta \sin\phi & \rho \cos\theta \cos\phi \\ \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\theta \sin\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi \end{vmatrix}$$

$$= \rho^2 \left[\cos\phi \left(-\sin^2\theta \sin\phi \cos\phi - \cos^2\theta \sin\phi \cos\phi \right) - \sin\phi \left(\cos^2\theta \sin^2\phi + \sin^2\theta \sin^2\phi \right) \right)$$

$$= \rho^2 \left[-\sin\phi \cos^2\phi \underbrace{\left(\sin^2\theta + \cos^2\theta \right)}_{1.} - \sin^3\phi \underbrace{\left(\cos^2\theta + \sin^2\theta \right)}_{=1} \right) \Big].$$

$$= \rho^2 \sin\phi \underbrace{\left(\sin^2\phi + \cos^2\phi \right)}_{1.} = \rho^2 \sin\phi .$$