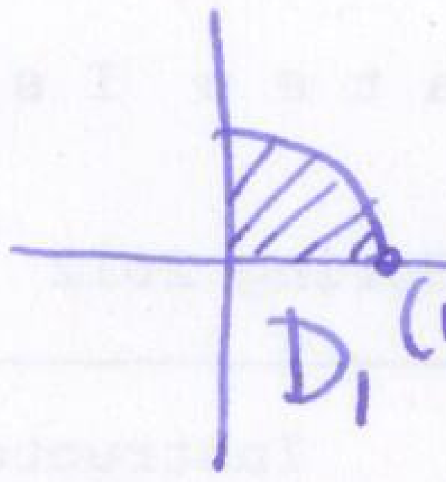
§15.4 Integration in polar, cylindrical and spherical coordinates

recall: double integrals over regions

$$\iint_D f(x,y)\,dA$$



some regions hard to describe in cartesian coordinates, but easy to describe in polars, e.g. $D_1: 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1$

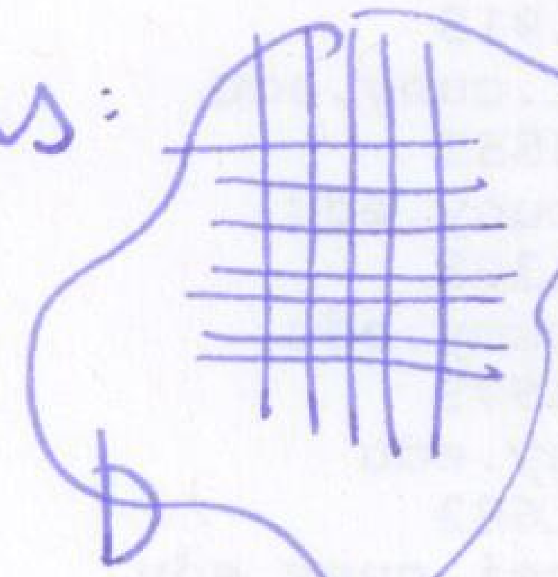
$D_2: 0 \le \theta \le 2\pi, 1 \le r \le 2.$

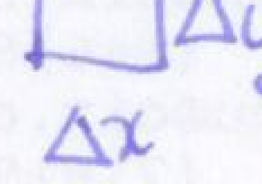
useful fact: in polar coordinates $\iint_D f(x,y)\,dA = \iint_D f(r,\theta)\, r\,dr\,d\theta$ $\quad f(x(r,\theta), y(r,\theta))$

i.e. $dA = dx\,dy$ in cartesians

$\boxed{dA = r\,dr\,d\theta}$ in polars. Warning: do not forget the r.

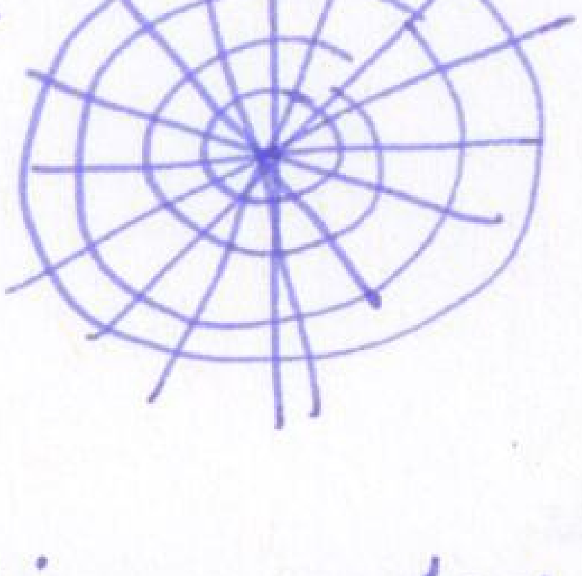
idea: cartesians:





area $\Delta A = \Delta x \Delta y$.

polars:



"squares" of different size

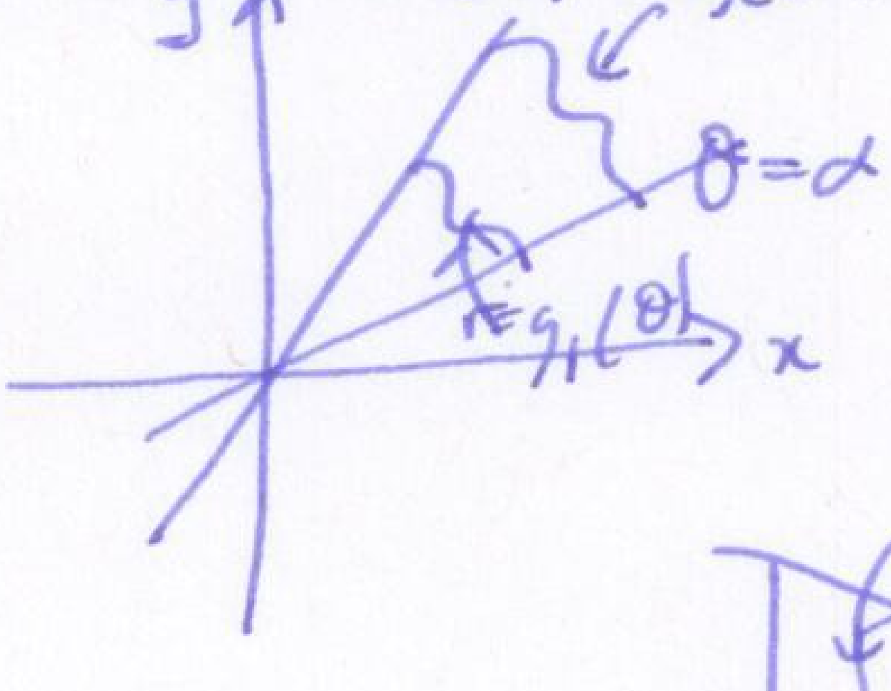
area $\Delta A \simeq r\Delta r \Delta\theta$

area of wedge

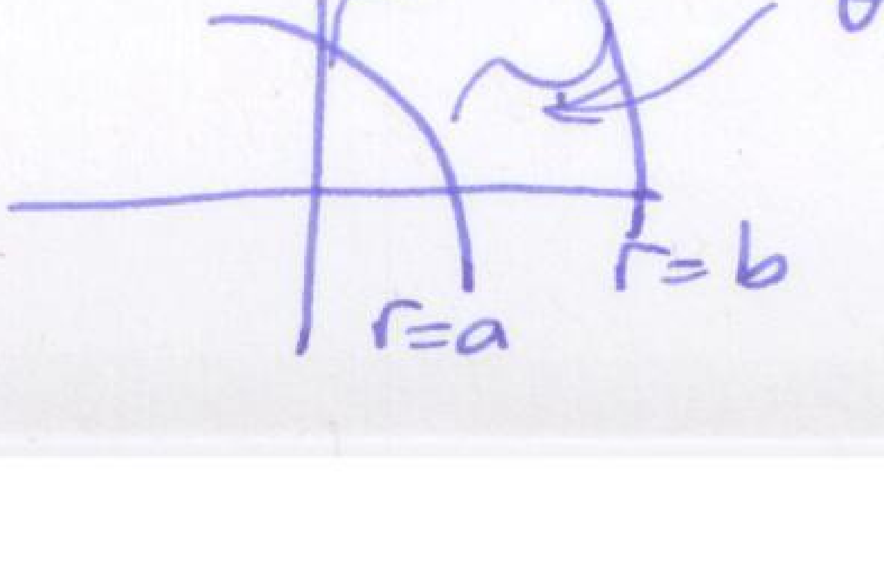
area of this is $(r+\Delta r)\Delta\theta - r\Delta\theta$
$= r\Delta r\Delta\theta$

Describing regions

$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$



$$\int_a^b \int_{g_1(r)}^{g_2(r)} f(r,\theta)\, r\,dr\,d\theta$$



Example Integrate $f(x,y) = xy$ over the upper half disc of radius 4.

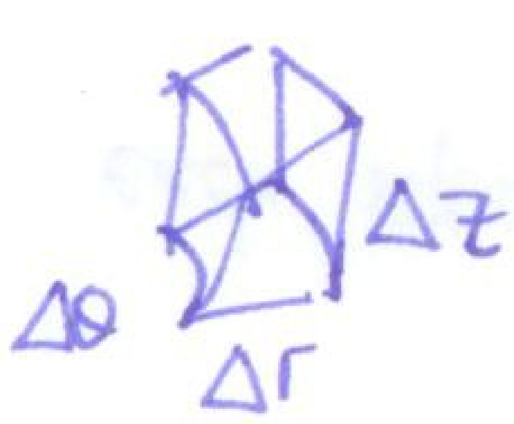
$$\iint_D f(x,y)\,dA \qquad \int_0^{\pi}\int_0^4 r^2\cos\theta\sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi} \tfrac{1}{2}\sin 2\theta \underbrace{\int_0^4 r^3\,dr}_{\left[\frac{1}{4}r^4\right]_0^4 = 4^3} d\theta = 32\left[-\tfrac{1}{2}\cos 2\theta\right]_{\pi}^{0} = 0.$$

Cylindrical coordinates $\iiint_R f(x,y,z)\,dV$ cartesian coords.

$\Delta V = \Delta x \Delta y \Delta z$

so $dV = dx\,dy\,dz$.



$\Delta V \simeq r\Delta r\Delta\theta\Delta z$

so $\iiint_R f(x(r,\theta), y(r,\theta), z)\, r\,dr\,d\theta\,dz$. $\boxed{dV = r\,dr\,d\theta\,dz}$

Spherical coords. $\boxed{dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$

so $\iiint_R f(x,y,z)\,dV = \iiint_R f(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi))\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$.