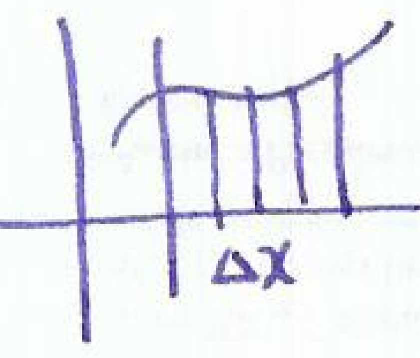
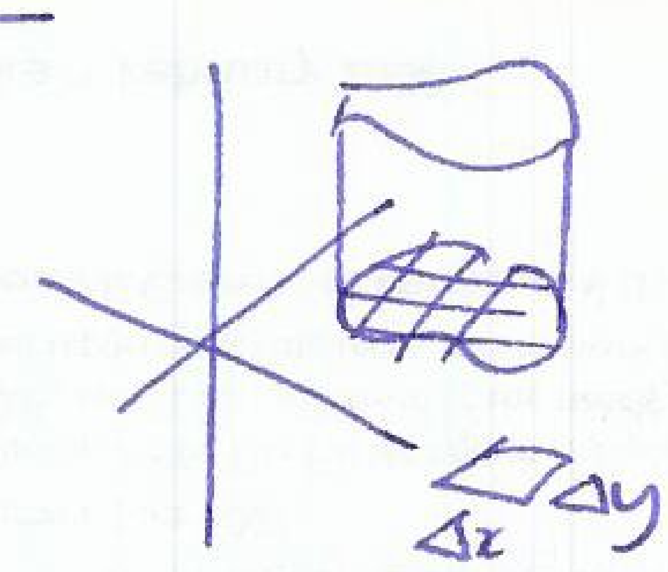
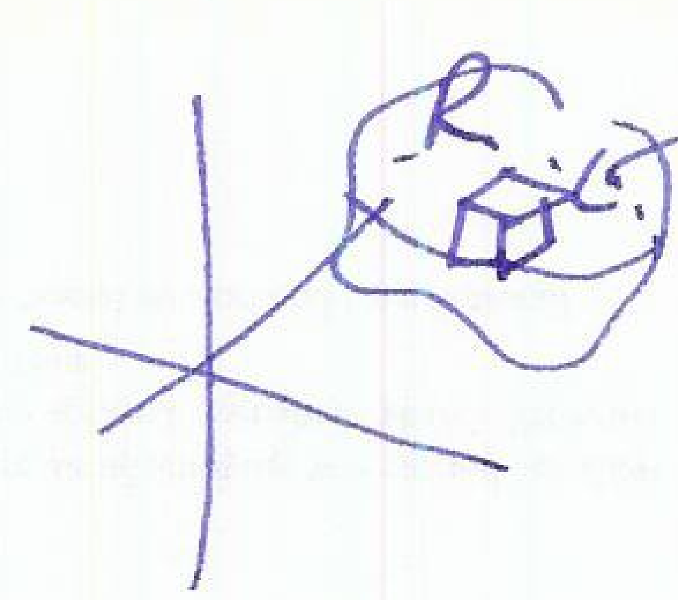
§15.3 Triple integrals

1d: 

2d: 

3d: 

divide into boxes 

$\sum f(x_i)\Delta x$ $\qquad$ $\sum f(x_i,y_i)\Delta x\Delta y$ $\qquad$ $\sum f(x_i,y_i,z_i)\Delta x\Delta y\Delta z.$

notation $\iiint_R f(x,y,z)\,dV$

Example integrate $f(x,y,z) = x^2e^{y+3z}$ over the box $\underset{x}{[1,2]}\times\underset{y}{[3,4]}\times\underset{z}{[0,1]}$.

$$\int_0^1\int_3^4\underbrace{\int_1^2 x^2e^{y+3z}\,dx}\,dy\,dz$$

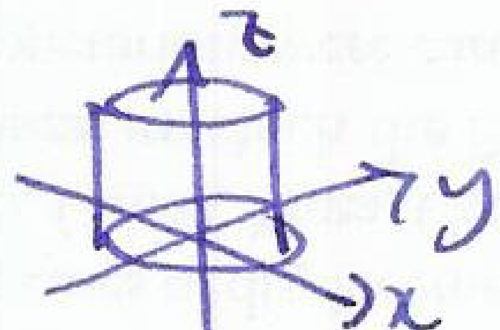
inner integral $= e^{y+3z}\int_1^2 x^2\,dx = e^{y+3z}\left[\frac{1}{3}x^3\right]_1^2 = \frac{7}{3}e^{y+3z}.$

$$\int_0^1\underbrace{\int_3^4 \frac{7}{3}e^{y+3z}\,dy}\,dz$$

inner integral $= \frac{7}{3}e^{3z}\int_3^4 e^y\,dy = \frac{7}{3}e^{3z}\left[e^y\right]_3^4 = \frac{7}{3}(e^4-e^3)e^{3z}.$

$$\int_0^1 \frac{7}{3}(e^4-e^3)e^{3z}\,dz = \frac{7}{3}(e^4-e^3)\left[\frac{1}{3}e^{3z}\right]_0^1 = \frac{7}{9}(e^4-e^3)(e^3-1).$$

Describing regions $\qquad$ Region $R \leftrightarrow$ limits in terms of x,y,z in some order.

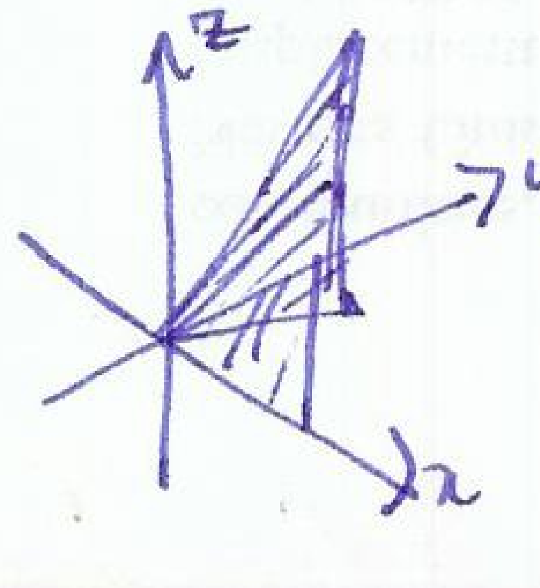
$R=$ cylinder 

$0\le z\le 1$

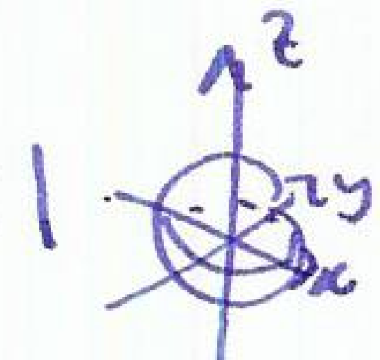
$x^2+y^2\le 1$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\int_0^1 f(x,y,z)\,dz\,dy\,dx.$$

$R=$ region between planes $z=x+y$ and $z=3x+5y$ over the triangle 

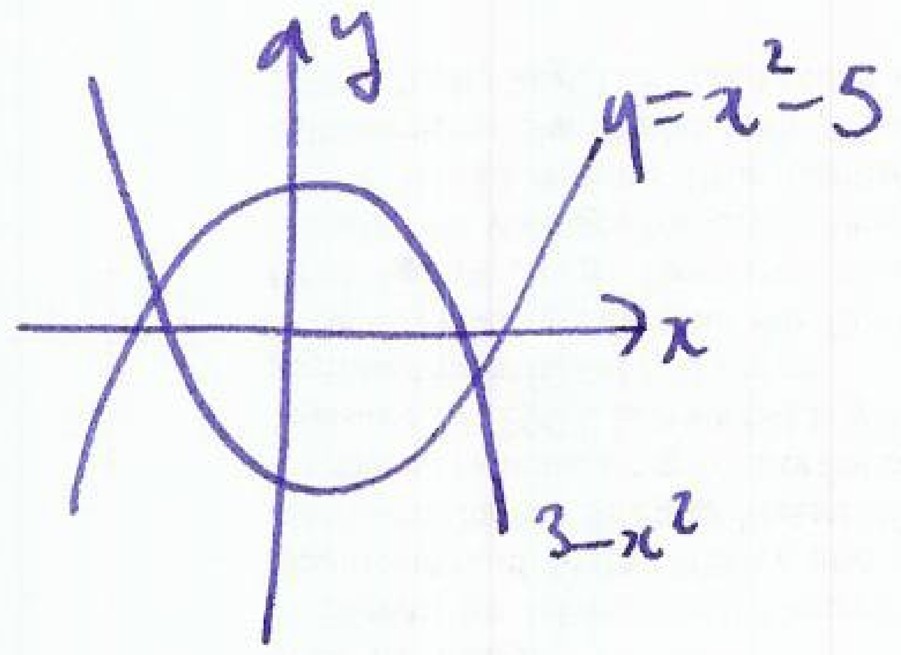


$$\int_0^1\int_0^{1-x}\int_{x+y}^{3x+5y} f(x,y,z)\,dz\,dy\,dx$$

$R=$ sphere $x^2+y^2+z^2\le 1$ 

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{+\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

Example find the area of the region between $y = 3-x^2$ and $y = x^2-5$.



find intersections: $3-x^2 = x^2-5$

$2x^2 = 8 \Rightarrow x = \pm 2$

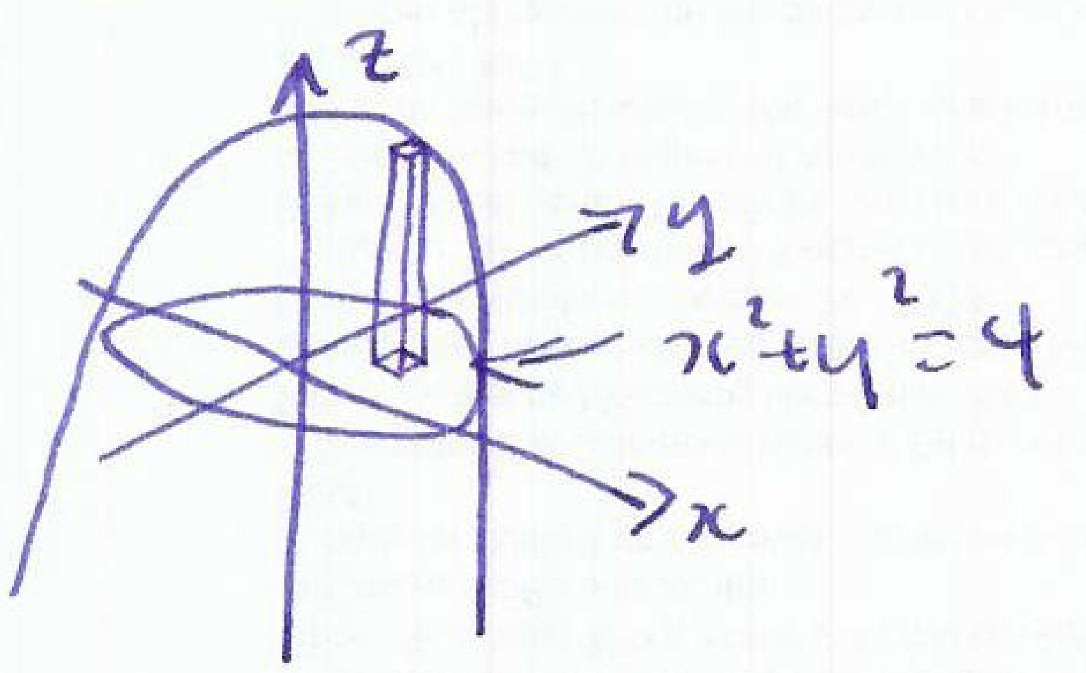
$$\int_{-2}^{2}\int_{x^2-5}^{3-x^2} 1 \, dy\,dx = \qquad [y]_{x^2-5}^{3-x^2} = (3-x^2)-(x^2-5)$$

$$= 8-2x^2.$$

$$\int_{-2}^{2} 8-2x^2 \, dx = \left[8x - \tfrac{2}{3}x^3\right]_{-2}^{2}$$

$$= 16 - \frac{16}{3} + 16 - \frac{16}{3}$$

Example find the volume of the region below $z = 4-x^2-y^2$ and above the xy-plane.



can do this in two ways.

$$\iiint_R 1 \, dV = \text{volume}.$$

or $$\iint_D 4-x^2-y^2 \, dA.$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} 1 \, dz\,dy\,dx \qquad [z]_0^{4-x^2-y^2} = 4-x^2-y^2.$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} 4-x^2-y^2 \, dy\,dx \qquad \left[y(4-x^2) - \tfrac{1}{3}y^3\right]_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}$$

$$= (4-x^2)^{3/2} - \frac{1}{3}(4-x^2)^{3/2} - \left(-(4-x^2)^{3/2} + \frac{1}{3}(4-x^2)^{3/2}\right)$$

$$= \frac{4}{3}(4-x^2)^{3/2}.$$

$$\int_{-2}^{2} \frac{4}{3}(4-x^2)^{3/2} \, dx$$

try: $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$.

$$\int_{-\pi/2}^{\pi/2} \frac{4}{3}(4-4\sin^2\theta)^{3/2} \cdot 2\cos\theta \, d\theta = \frac{8}{3}\int_{-\pi/2}^{\pi/2} 4^{3/2} \cos^3\theta \cos\theta \, d\theta$$

$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$
$$\cos^2\theta = \tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}.$$

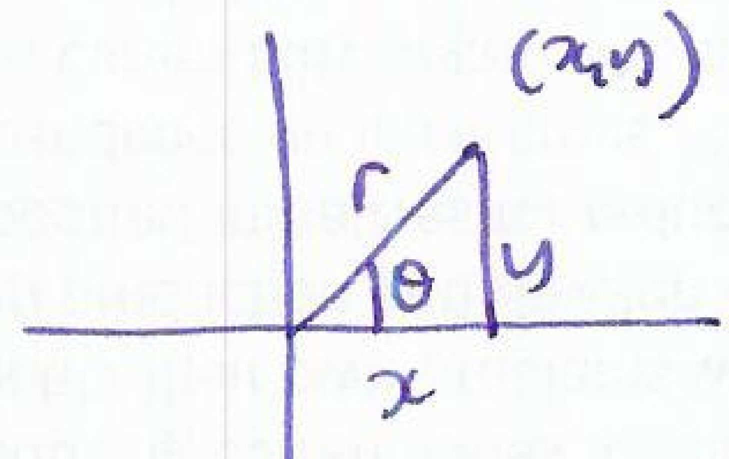
$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2} \left(\tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}\right)^2 d\theta = \frac{64}{3}\int_{-2}^{2} \frac{1}{4}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right) d\theta$$

$$= \frac{16}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos 4\theta + \frac{1}{2} + 2\cos 2\theta + 1 \, d\theta$$

$$= \frac{16}{3}\left[-\frac{1}{8}\sin 4\theta + \frac{3\theta}{2} - \sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{16}{3}\cdot\frac{3}{2}\cdot 2\frac{\pi}{2} = 8\pi.$$

§12.7 Cylindrical and spherical coordinates.

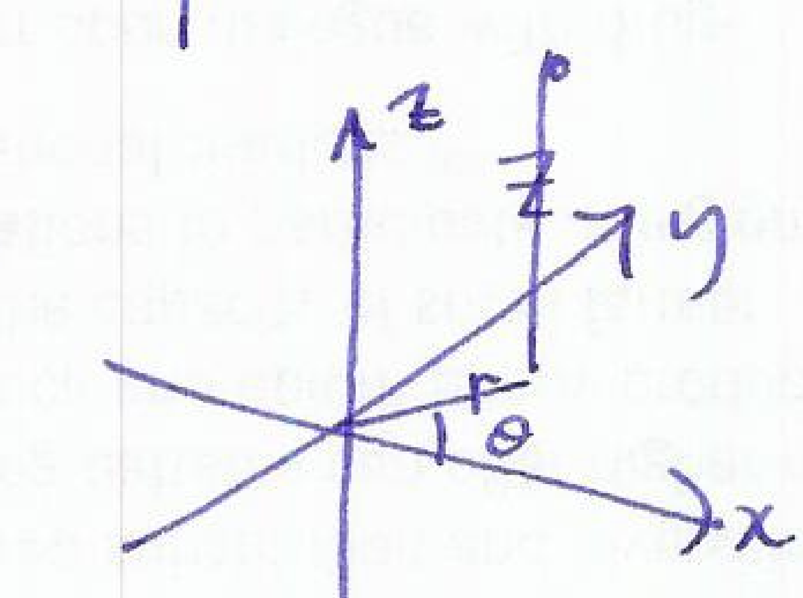
Polar coordinates:



$r^2 = x^2 + y^2$ $\quad x = r\cos\theta$

$\tan\theta = y/x$ $\quad y = r\sin\theta.$

Cylindrical coordinates:

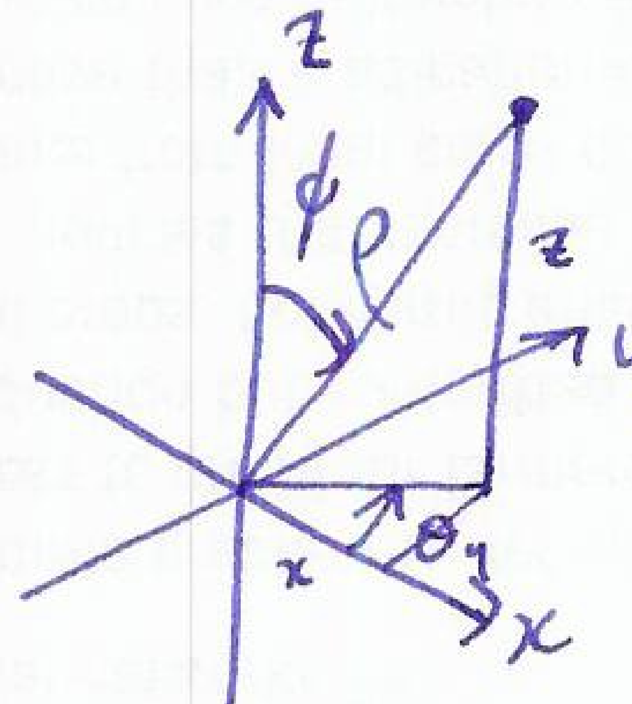


$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \theta = \tan^{-1}(y/x).$

$z = z \qquad z = z.$

Spherical coordinates

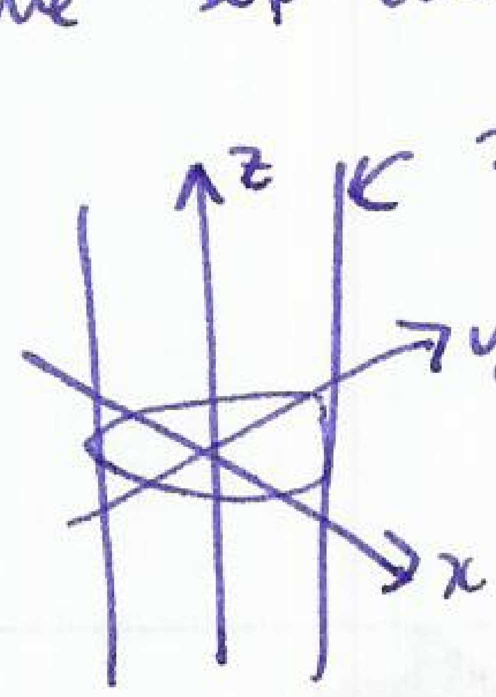


$x = \rho\cos\theta\sin\phi \qquad \rho^2 = x^2+y^2+z^2$

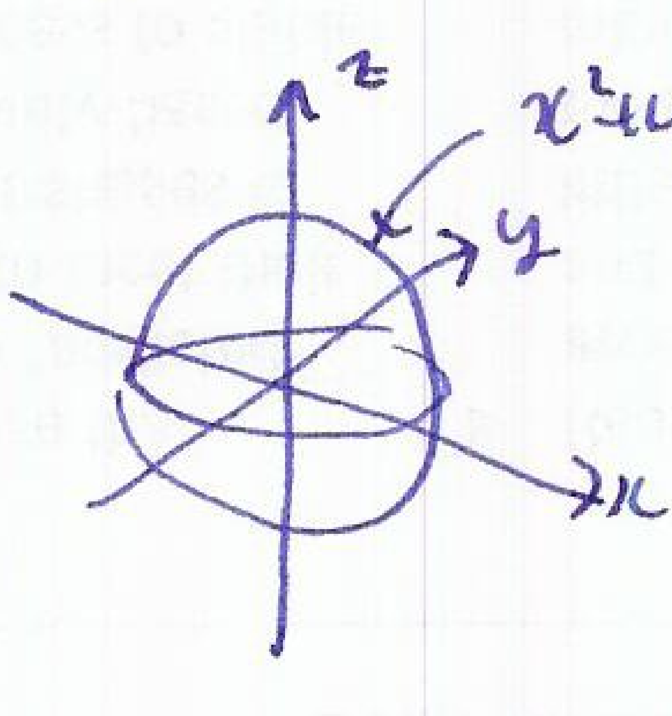
$y = \rho\sin\theta\sin\phi \qquad \theta = \tan^{-1}(y/x).$

$z = \rho\cos\phi \qquad \phi = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right)$

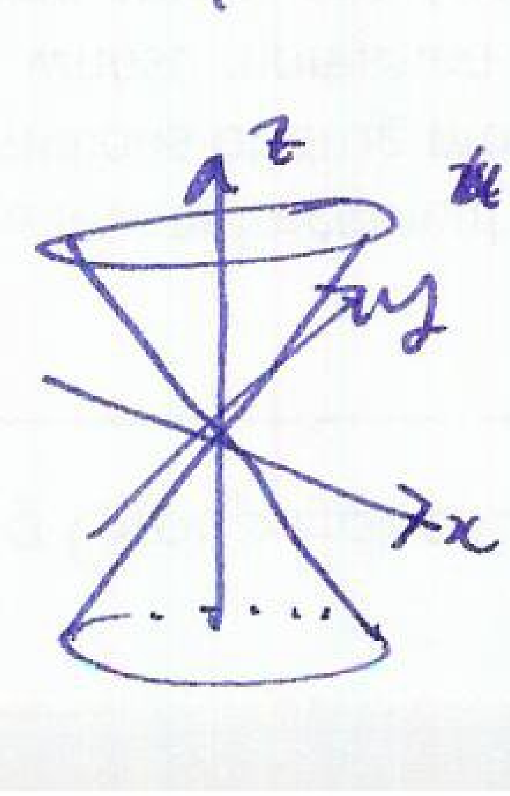
Some sets are easier to describe in these coordinates.



$x^2+y^2 = 1 \iff r = 1$ in cylindrical coordinates.



$x^2+y^2+z^2 = 1 \iff \rho = 1$ in spherical coordinates.



$z^2 = x^2+y^2 \iff z^2 = r^2.$ in cylindricals.