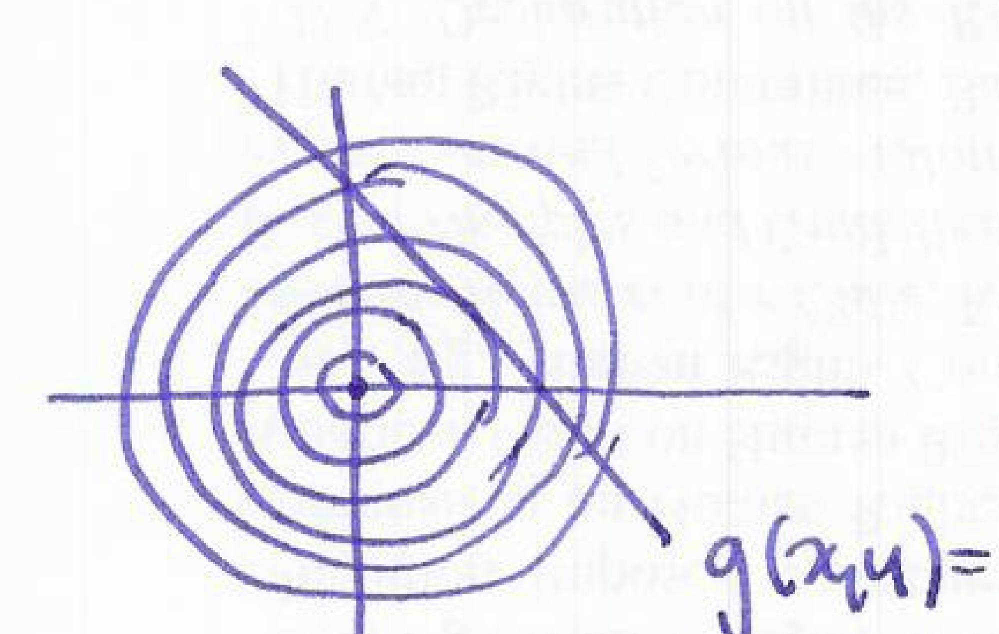
§14.8 Lagrange multipliers

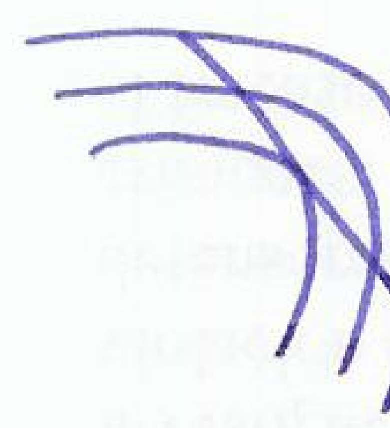
optimization with constraint, i.e. max $f(x,y)$ subject to $g(x,y)=0$

Example maximize $f(x,y)=\sqrt{x^2+y^2}$ subject to $\underbrace{x+y-4}_{g(x,y)}=0$



consider level sets of $f(x,y)$, i.e. $f(x,y)=c$.

key point: at the extreme values of $f(x,y)$ on $g(x,y)=0$, the level sets of f are parallel to g.



so we want to look for points where level sets of f parallel to level sets of g

i.e. solve $\nabla f=\lambda\nabla g$ $(\lambda\neq 0)$

Example: $\nabla f=\left\langle \frac{1}{2}(x^2+y^2)^{-1/2}2x,\ \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y\right\rangle=\lambda\nabla g=\lambda\langle 1,1\rangle.$

$$\Rightarrow \left.\begin{aligned}\frac{x}{\sqrt{x^2+y^2}}&=\lambda \quad ①\\ \frac{y}{\sqrt{x^2+y^2}}&=\lambda \quad ②\end{aligned}\right\}\quad \frac{①}{②}\Rightarrow \frac{x}{y}=1\Rightarrow x=y.$$

now plug into $g(x,y)=0$:

$x+x-4=0\Rightarrow x=2$, $y=2$.

Summary want to max/min f given $g=0$

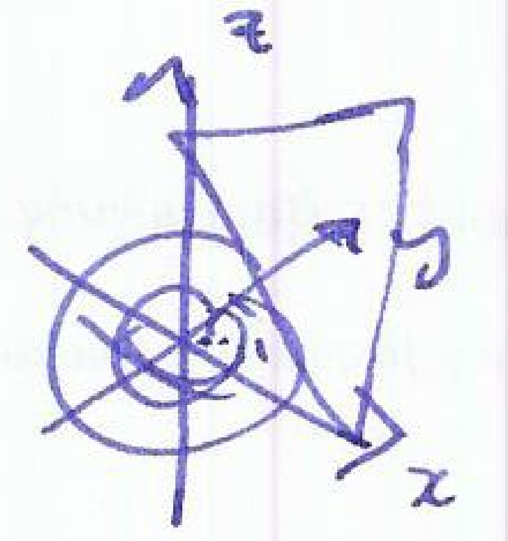
① solve $\nabla f=\lambda\nabla g$, $g=0$.

② ~~plug that~~ Example find the point on the plane $x+2y+3z=4$ closest to the origin.

i.e. min $\underbrace{x^2+y^2+z^2}_{f(x,y,z)}$ subject to $x+2y+3z=4$

solve $\nabla f = \lambda \nabla g$ $\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 1, 2, 3 \rangle$



$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= 2\lambda \\ 3z &= 3\lambda \end{aligned}\right\} \quad \left.\begin{aligned} x &= \lambda/2 \\ y &= \lambda \\ z &= 3\lambda/2 \end{aligned}\right\}$$

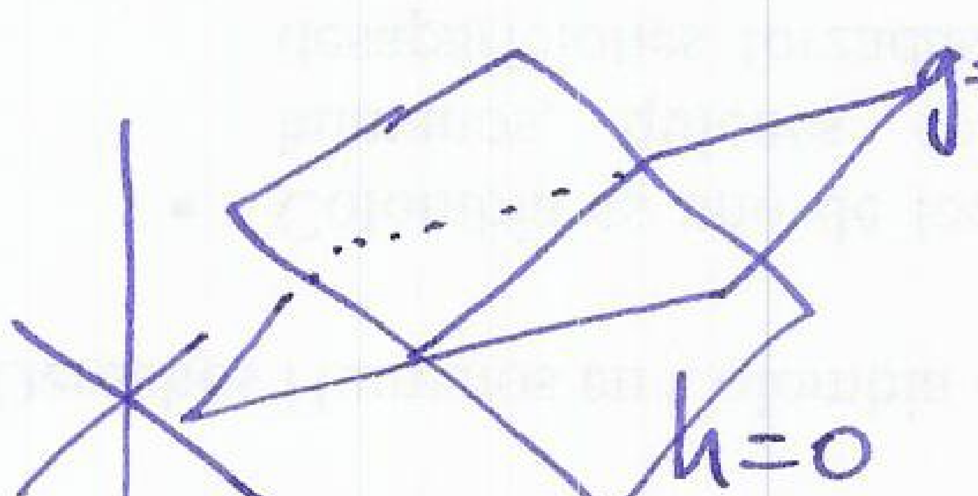
$x + 2y + 3z = 4.$ $\frac{\lambda}{2} + 2\lambda + \frac{9\lambda}{2} = 4.$ $\lambda = 4/7$

so point is $\langle \frac{2}{7}, \frac{4}{7}, 6/7 \rangle$

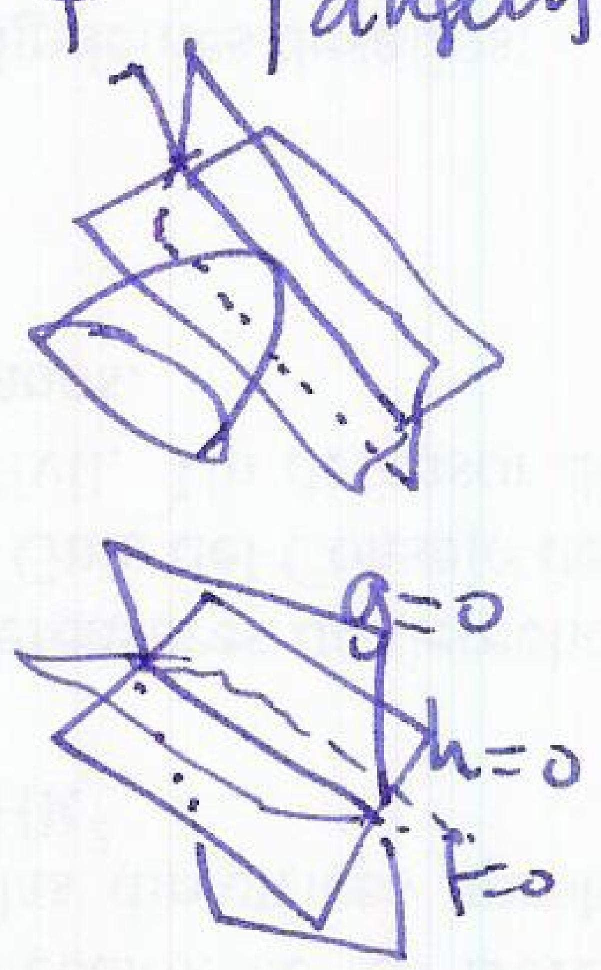
<u>Example</u> minimize $f(x,y,z) = x^2y^2 + z^2$ subject to $x + y = 2$ <u>and</u> $y + z = 4$

solve $\nabla f = \lambda \nabla g + \mu \nabla h$

<u>why?</u>



extreme point when level set of f tangent to $g=0 \cap h=0$.



i.e. tangent directions to $g=0$ and $h=0$ ⊂ tangent plane to $f=c$

span $g=0 \cap h=0 \subset$ 3rd plane

$\Rightarrow \nabla f = \lambda \nabla g + \mu \nabla h.$

$\nabla f = \langle 2xy^2, 2x^2y, 2z \rangle$

$\nabla g = \langle 1, 1, 0 \rangle$

$\nabla h = \langle 0, 1, 1 \rangle$

$$\left.\begin{aligned} 2xy^2 &= \lambda \\ 2x^2y &= \lambda + \mu \\ 2z &= \mu \end{aligned}\right\} \quad 2x^2y = 2xy^2 - 2z. \quad z = \frac{\mu}{2}$$

$$\left.\begin{aligned} x + y &= 2 \\ y + z &= 4 \end{aligned}\right\} \quad \begin{aligned} y &= 2 - x \\ z &= 4 - y = 2 + x \end{aligned}$$

$2x^2(2-x) = 2x(2-x)^2 - 2(2+x)$

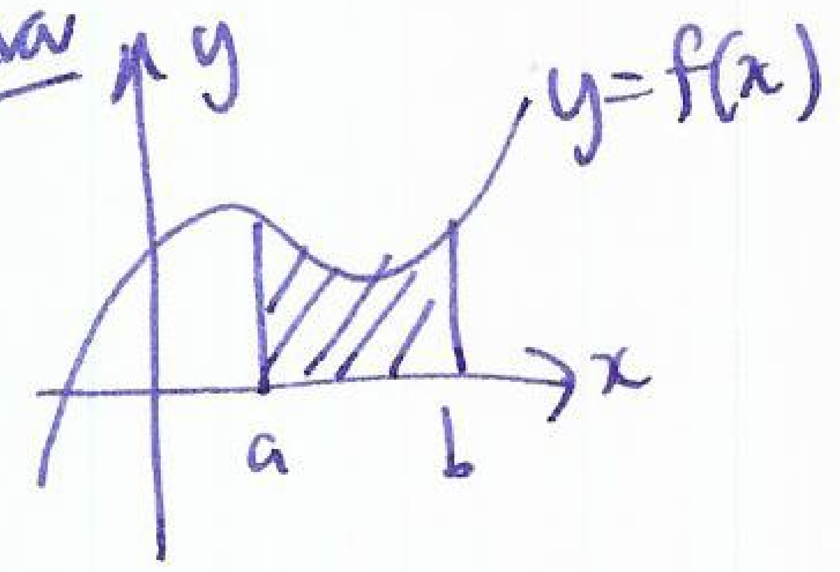
↑ cubic can't solve exactly

$x \simeq -0.3$ (unique solution)

. then $y \simeq 2.3$

$z \simeq 1.7$.

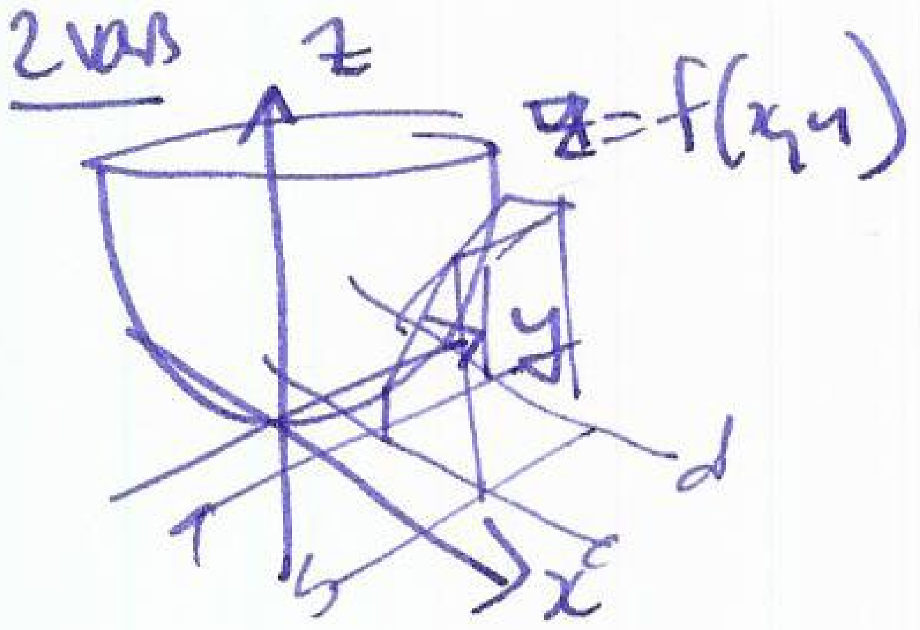
§14.1 Integration in many variables

1var 

$\int_a^b f(x)\,dx$ = area under curve between a and b.

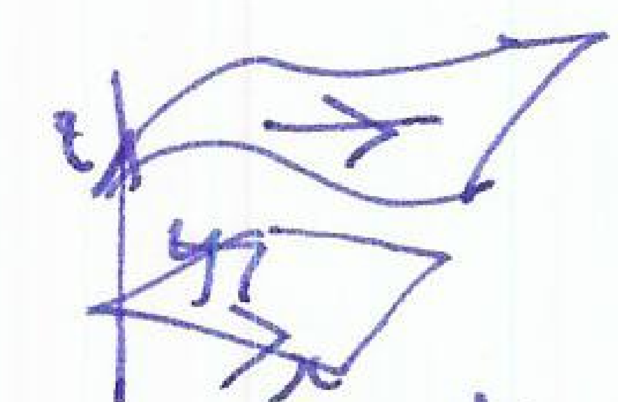
↑ compute this by finding an antiderivative $F(x)$ s.t. $F'(x) = f(x)$

then $\int_a^b f(x)\,dx = F(b) - F(a)$.

2 vars 

$\int_a^b \int_c^d f(x,y)\,dy\,dx$ = area under ~~curve~~ surface & ove region $a \le x \le b$, $c \le y \le d$

↑ compute this by doing iterated integrals.



recall : $\frac{\partial f}{\partial x}$ = "diff wrt x keeping y constant" – slope in x-direction

$\int_a^b f(x,y)\,dx$ = "integrate wrt x keeping y constant"

Example. $\int_0^1 xy\,dx = \left[\frac{1}{2}x^2 y\right]_0^1 = \frac{1}{2}y$ note: there should be no x's remaining

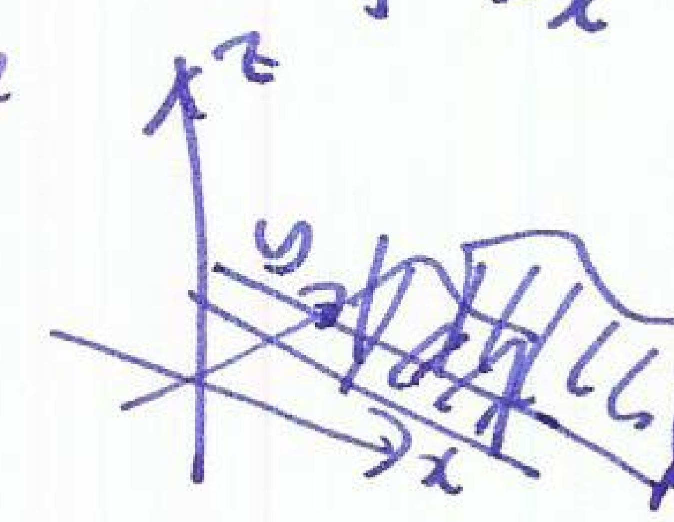
The limits can depend on y!

$\int_y^{y^2} xy\,dx = \left[\frac{1}{2}x^2 y\right]_y^{y^2} = \frac{1}{2}y^5 - \frac{1}{2}y^3$

Warning: can't have x's in the limits.

Q-what does this mean: $\int_0^1 xy\,dy$ means area between 0 and 1

$\int_y^{y^2} xy\,dy$ mean area between y and y^2



Double integrals

$\underbrace{\int_0^1 \int_0^1}_{\text{describes region}} xy\,dx\,dy = \int_0^1 \frac{1}{2}y\,dy = \left[\frac{1}{4}y^2\right]_0^1 = \frac{1}{4}$

$0 \le y \le 1$

$0 \le x \le 1$