3) if $D=0$ no information.

Why does this work?

Quadratic approximation is $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-a) + [x-a \; x-b] \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} a-b \\ y-b \end{bmatrix}$

quadratic terms are: $f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2$

this is a quadratic form, like x^2+y^2.  $-(x^2+y^2)$  xy or x^2-y^2 saddles. 

up to change of coords a symmetric matrix looks like

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \leftrightarrow \lambda_1 x^2 + \lambda_2 y^2$.

$\lambda_1, \lambda_2 > 0$ min
$\lambda_1, \lambda_2 < 0$ max
different signs saddle
0 don't know

$D =$ det of matrix $= \lambda_1 \lambda_2$.

so $D > 0$ same sign
$D < 0$ different signs.

Example find the critical points of $f(x,y) = (x^2+y^2)e^{-2x} = x^2e^{-2x} + y^2e^{-2x}$

$f_x = 2xe^{-2x} + x^2 \cdot -2e^{-2x} - 2y^2e^{-2x}$

$f_y = 2ye^{-2x}$

solve ① $f_x = 0$
② $f_y = 0$

②: $e^{-2x} \neq 0 \Rightarrow y = 0$

①: $2xe^{-2x} - 2x^2e^{-2x} = 0$. $e^{-2x} \neq 0$.

$\Rightarrow 2(x - x^2) = 0 \quad x = 0, 1$. $(0,0)$ $(1,0)$

$f_{xx} = 2x \cdot -2e^{-2x} + 2e^{-2x} + 2x \cdot -2e^{-2x} + x^2 \cdot 4e^{-2x} + 4y^2e^{-2x}$.

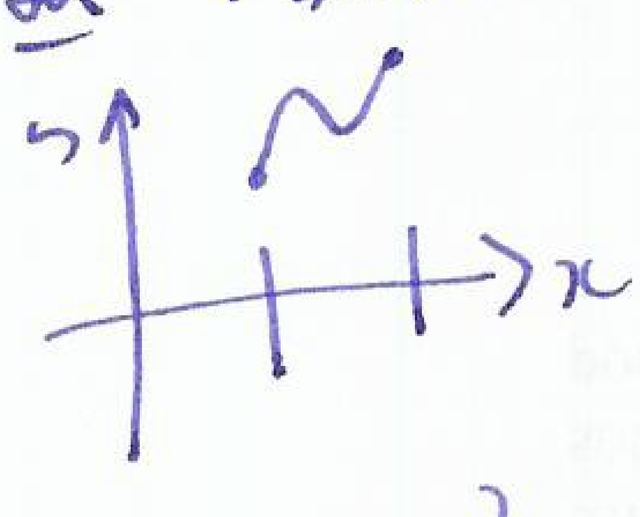
$f_{xy} = -4ye^{-2x}$

$f_{yy} = 2e^{-2x}$

$D = f_{xx} f_{yy} - (f_{xy})^2 \ldots$

Global extrema (absolute max/min)

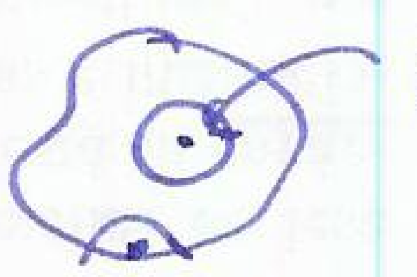
1d $f: [\,] \to \mathbb{R}$ a continuous function on a closed interval has an absolute max and min.



2d: $D \subset \mathbb{R}^2$ $f: D \to \mathbb{R}$.

D is bounded if $D \subset$ disc of radius R about $(0,0)$.

$x \in D$ is interior point if a small ball around $x \in D$


interior point
← boundary point.

otherwise x is a boundary point

interior of D = union of all interior points
Boundary of D = union of all boundary points

D is closed if D contains all its boundary points
D is open if every point is an interior point

Example $[a,b]$ (a,b) $\mathbb{R}$ closed and open.

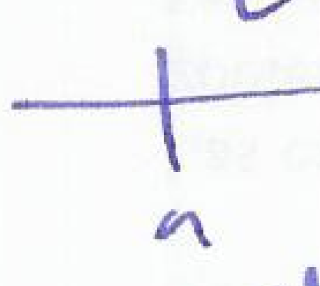
a b closed, open

$[a,b)$ neither closed nor open.

Thm $f(x,y)$ cts on $D \subseteq \mathbb{R}^2$ closed, bounded.
then · $f(x,y)$ has absolute max and min on D
& and · the extreme values occur at either critical points in the interior or at points on the boundary.

Example find the abs maximum/min of $f(x,y) = xy$ on the

diamond
square
circle


$(1,1)$
$(-1,1)$
$x^2+y^2=1$

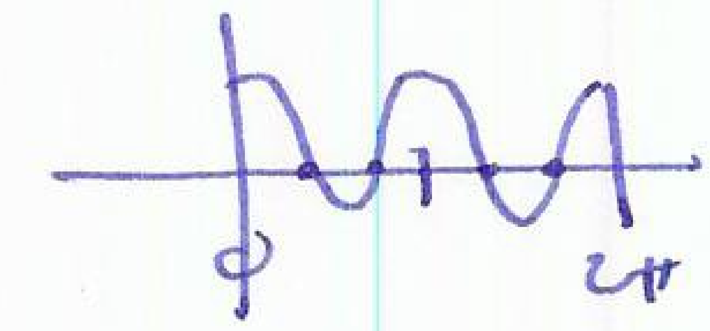
① find critical points.
$f_x = y$, $f_y = x$ } $(0,0)$ only critical point.

② find max on boundary

parameterize boundary: $(\cos\theta, \sin\theta)$ $0 \le \theta \le 2\pi$

$f(\theta) = f(x(\theta), y(\theta)) = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$

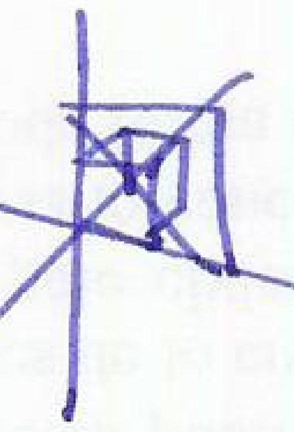
$f'(\theta) = \cos 2\theta$ $\quad f'(\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$



corresponds to points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$.

<u>Example</u> find the box of max volume bounded by the coordinate planes and the plane $x+y+z=1$



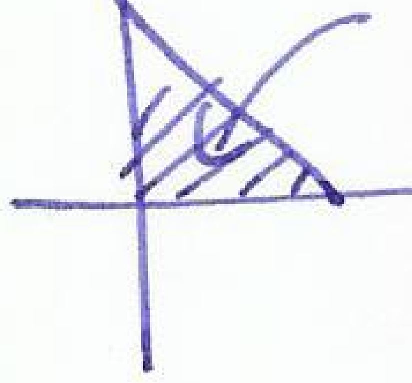
sides of box x, y, z, with $x+y+z=1$.

volume of box $V = xyz$

so sides $x, y, z = 1-x-y$

volume $V = xy(1-x-y) \leftarrow$ so max this on $\{(x,y) \mid x \ge 0, y \ge 0, z \ge 0, 1-x-y \ge 0, x+y \le 1\}$.

$= xy - x^2y - xy^2$



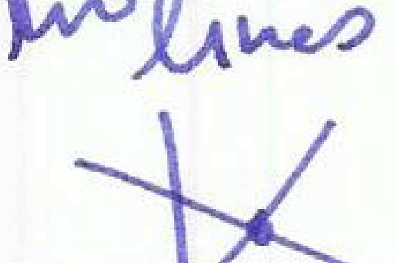
max V on this region.

(1) find critical points $V_x = y - 2xy - y^2$ (1)

$V_y = x - x^2 - 2xy$ (2)

solve $\left.\begin{array}{l} V_x = 0 \\ V_y = 0 \end{array}\right\}$ (1): $y - 2xy - y^2 = 0 \Rightarrow x = \frac{y-y^2}{2y} = \frac{1}{2}(1-y)$

(2): $x - x^2 - 2xy = 0 \Rightarrow y = \frac{x-x^2}{2x} = \frac{1}{2}(1-x)$ } two lines



intersect at: $\left.\begin{array}{l} 2x+y=1 \\ x+2y=1 \end{array}\right\}$ $2(1)-(2)$: $3y = -1$ $y = 1/3$ $x = 1/3$. $\left(\frac{1}{3}, \frac{1}{3}\right)$ critical point

(2) check max on boundary: 3 edges

$x=0$ $0 \le x \le 1$

$V=0$

$y=0$ $0 \le y \le 1$

$V=0$

$(t, 1-t)$ $0 \le t \le 1$ $\Leftrightarrow z=0$!

$V(t) = t(1-t)(1-t-(1-t))$

$= t(1-t) \cdot 0 = 0$.