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Example $f(x,y,z) = xy + z^3$

$f: \mathbb{R}^3 \to \mathbb{R}$

$$\left.\begin{aligned} x &= s+t \\ y &= s-t \\ z &= st \end{aligned}\right\} \quad \begin{aligned} g &: \mathbb{R}^2 \to \mathbb{R}^3. \\ (s,t) &\mapsto (x,y,z) \end{aligned}$$

so $f(g(s,t))$ makes sense.

<u>Q</u>. find $\frac{\partial f}{\partial s}$:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}.$$

$$= y \cdot 1 + x \cdot 1 + 3z^2 \cdot t$$

$$= (s-t) + (s+t) + 3(st)^2 \cdot t.$$

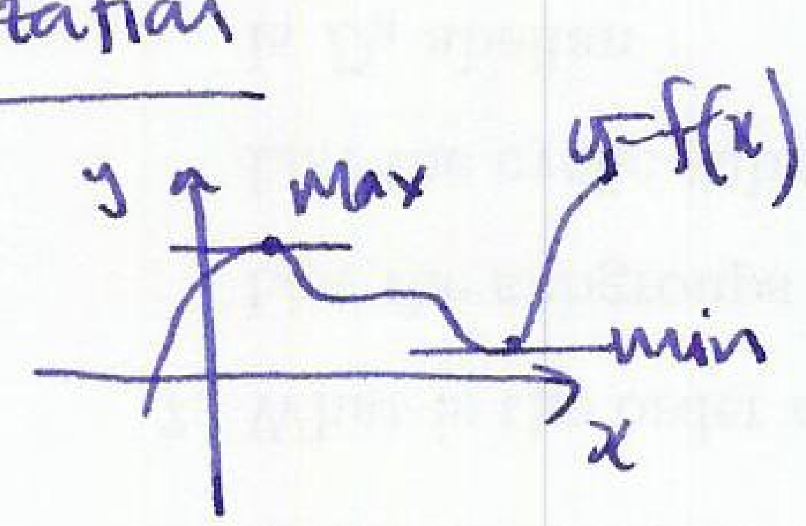
<u>mnemonic</u> to find $\frac{\partial f}{\partial y_i}$ $f(x_1, x_2 \dots x_n)$ $x_i(y_1, y_2 \dots y_m)$.

then $$\frac{\partial f}{\partial y_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_i} + \dots + \frac{\partial f}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_i}$$

"differentiate f wrt all variables".

<u>§14.7 Optimization</u>

<u>recall</u> 1 var:

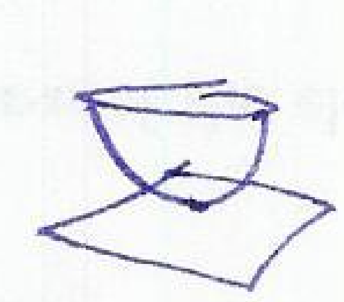


max, min $\Rightarrow \frac{dy}{dx} = 0$.

$\frac{dy}{dx} \not\Rightarrow$ max, min.

<u>2 vars</u> local max

local min



$\Rightarrow$ flat tangent plane.

$z =$ const.

recall: tangent plane is $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

so flat tangent plane $\Rightarrow$ $f_x(a,b) = 0$ <u>and</u> $f_y(a,b) = 0$

warning $f_x(a,b) = 0$ and $f_y(a,b) = 0$ $\not\Rightarrow$ local max or min.

<u>Def</u> A <u>critical point</u> (a,b) is a point such that $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$.

or at least one of $f_x(a,b)$, $f_y(a,b)$ does not exist.

<u>Thm</u> If $f(x,y)$ has a local max or min at (a,b) then $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$

<u>Example</u> finding critical points.

$$f(x,y) = x^2 - 2xy + 2y^2 + 3y + 1$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x - 2y \quad ① \\ \frac{\partial f}{\partial y} &= -2x + 4y + 3 \quad ② \end{aligned}\right\} \text{ solve these } = 0$$

$① + ②$: $2y + 3 = 0$

$y = -3/2$

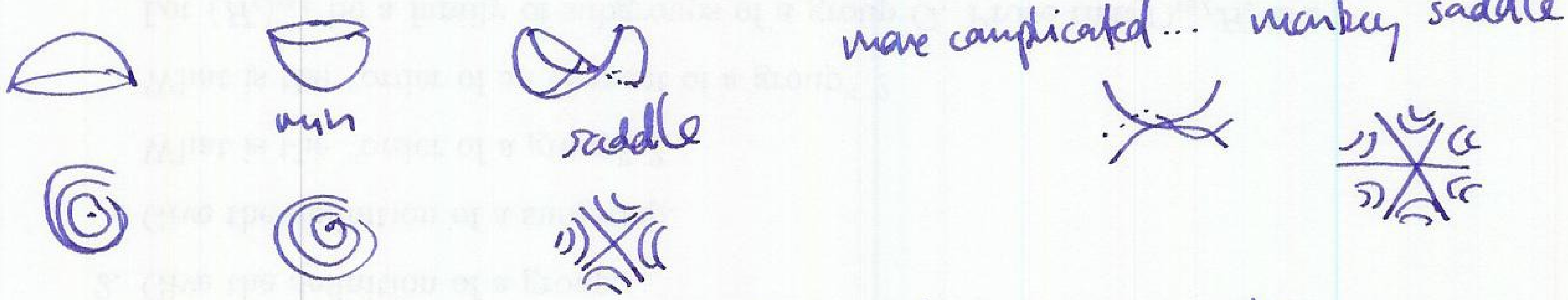
$x = -3/2$.

only one critical point at $(-3/2, -3/2)$

<u>Types of critical point</u>:

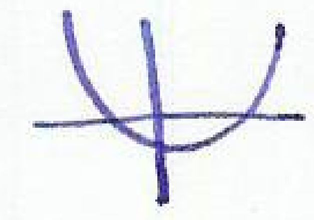
local max

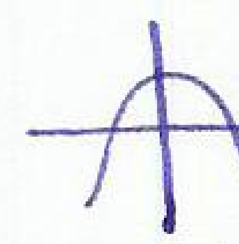
contour/level sets.



How to tell which one? Look at 2nd order/quadratic approximation.

<u>1 var</u>: $f(x) \approx f(a) + \underbrace{f'(a)(x-a)}_{=0 \text{ if critical point}} + \underbrace{\tfrac{1}{2}f''(a)(x-a)^2}_{} + \cdots$

$f''(a) > 0$  min

$f''(a) < 0$  max.

$f''(a) = 0$ no information!

<u>2 vars</u>:

<u>Thm</u> (2nd derivative test) Let (a,b) be a critical point for $f(x,y)$

Let $D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2$ then

1) if $D > 0$ then $f(a,b)$ is a <u>minimum</u> if $f_{xx}(a,b) > 0$
<u>maximum</u> if $f_{xx}(a,b) < 0$

2) if $D < 0$ then $f(a,b)$ is a <u>saddle</u>.