Defn $f(x,y)$ is <u>differentiable</u> at (a,b) if

- $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ exist
- $f(x,y)$ is locally linear at (a,b)

in this case the tangent plane is $z = L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Thm If $f_x(x,y), f_y(x,y)$ exist and are continuous close to (a,b) then $f(x,y)$ is differentiable at (a,b).

Bad example is $z^2 = x^2+y^2$ differentiable at $(0,0)$.

$z = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

not cts at $(0,0)$.

§14.5 Gradient

2d: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y)$

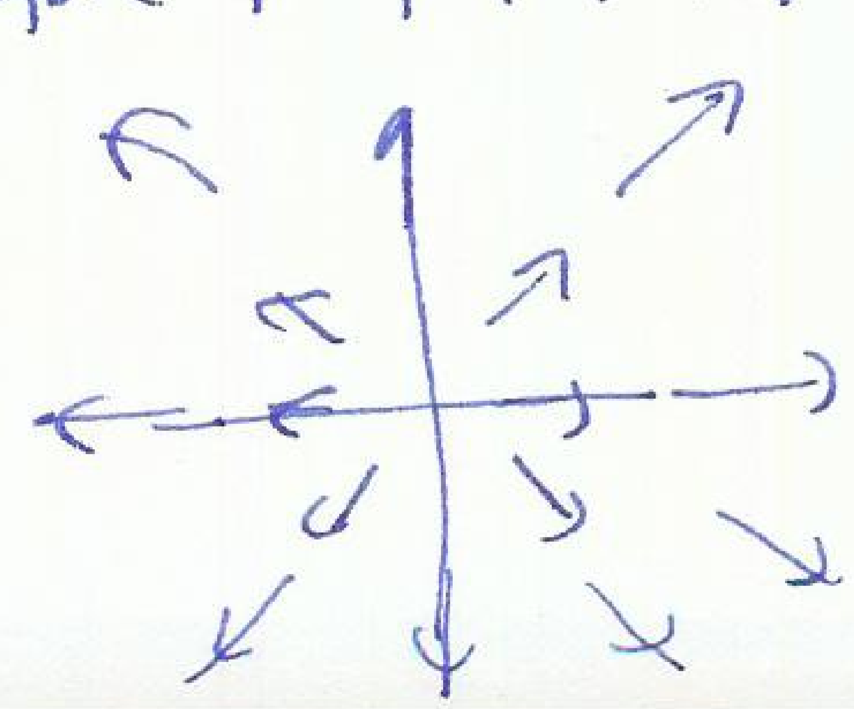
the <u>gradient</u> of f is $\underline{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$$\underline{\nabla} f(a,b) = \langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \rangle$$

3d: $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z)$ $\quad \underline{\nabla} f(a,b,c) = \langle \frac{\partial f}{\partial x}(a,b,c), \frac{\partial f}{\partial y}(a,b,c), \frac{\partial f}{\partial z}(a,b,c) \rangle$

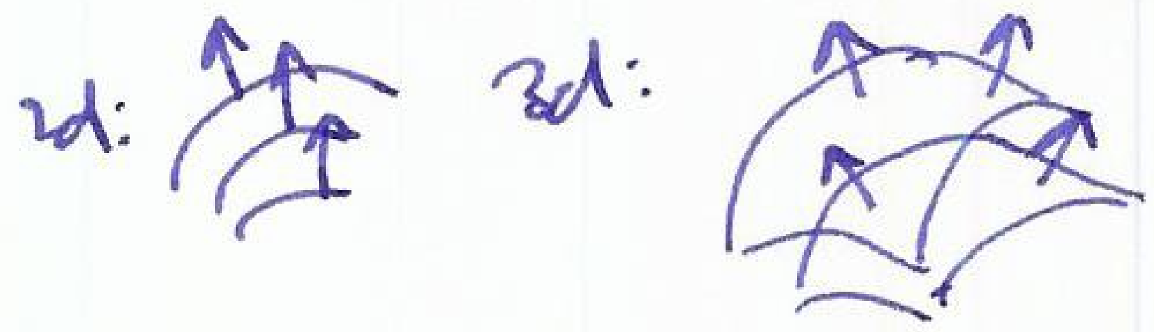
Note: ∇f: point $\to$ vector
$\mathbb{R}^2 \to \mathbb{R}^2$
$\mathbb{R}^3 \to \mathbb{R}^3$].

key facts:
- the gradient vector points in the direction of fastest rate of change
- $\|\nabla f\| =$ fastest rate of change.



Example $f(x,y) = x^2+y^2$ $\quad \nabla f = \langle 2x, 2y \rangle$

observation: the gradient vector is perpendicular to the level sets / contours.



Directional derivatives   useful properties

1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(cf) = c\nabla f$
3) product rule $\nabla(fg) = f\nabla g + g\nabla f$
4) chain rule: if $F(t)$ is differentiable then

$$\nabla\big(F(f(x,y,z))\big) = F'\big(f(x,y,z)\big)\nabla f$$

check this makes sense

$F\circ f$: $\mathbb{R}^3 \xrightarrow{f} \mathbb{R} \xrightarrow{F} \mathbb{R}$

$$\nabla\big(F(f(\underline{x}))\big) = \left< \frac{\partial}{\partial x}\big(F(f(\underline{x}))\big), \frac{\partial}{\partial y}\big(F(f(x,y,z))\big), \frac{\partial}{\partial z}\big(F(f(x,y,z))\big) \right>$$

$f(\underline{x}) = f(x,y,z)$
$$= \left< F'(f(\underline{x}))\cdot\frac{\partial f}{\partial x}, F'(f(\underline{x}))\frac{\partial f}{\partial y}, F'(f(\underline{x}))\cdot\frac{\partial f}{\partial z} \right>.$$

$$= F'(f(\underline{x}))\nabla f$$

Chain rule for paths

$f: \mathbb{R}^3 \to \mathbb{R}$ path $\underline{c}: \mathbb{R} \to \mathbb{R}^3$
$t \mapsto (x(t), y(t), z(t))$

composition: $c\circ f$ $\mathbb{R} \xrightarrow{c} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$

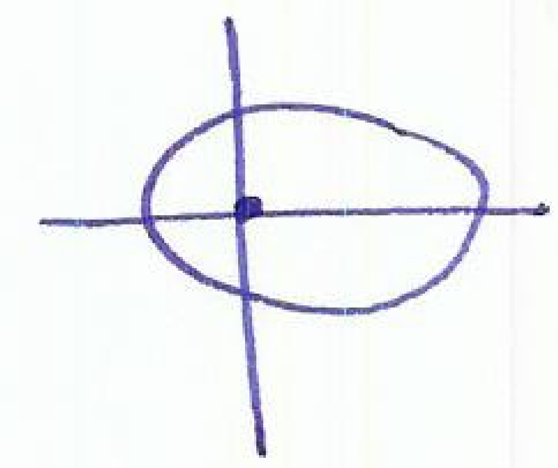
Thm $\dfrac{d}{dt} f(\underline{c}(t)) = \nabla f(\underline{c}(t)) \cdot \underline{c}'(t)$

~~Proof~~ in $\mathbb{R}^2$: $\dfrac{d}{dt}\big(f(\underline{c}(t))\big) = \left< \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right> \cdot \left< x'(t), y'(t) \right> = \dfrac{\partial f}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

<u>Example</u> temperature T varies with location like $T(x,y,z) = \frac{1}{x^2+y^2+z^2}$

if we move an ellipse $\frac{(x-3)^2}{25} + \frac{y^2}{16} = 1$ (focus at $(0,0)$, $(6,0)$)

parameterize ellipse : $\underline{c}(t) = (5\cos t + 3,\ 4\sin t,\ 0)$



$$T(\underline{c}(t)) = \frac{1}{(5\cos t+3)^2 + (4\sin t)^2 + (0)^2}$$

$$\frac{d}{dt}\big(T(\underline{c}(t))\big) = \underline{\nabla T} \cdot \underline{c}'(t) = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$= \left\langle -(x^2+y^2+z^2)^{-2} \cdot 2x,\ -(x^2+y^2+z^2)^{-2} \cdot 2y,\ -(x^2+y^2+z^2)^{-2} \cdot 2z \right\rangle \cdot \langle -5\sin t, 4\cos t, 0 \rangle$$

$$= -\frac{2}{(x^2+y^2+z^2)^2} \langle x, y, z \rangle \cdot \langle -5\sin t, 4\cos t, 0 \rangle$$

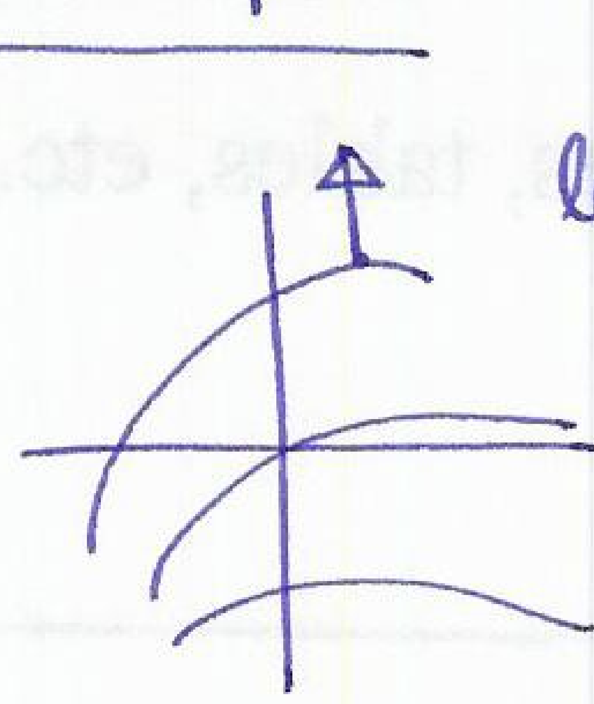
$$= \frac{-2}{(5\cos t+3)^2 + (4\sin t)^2} \langle 5\cos t + 3, 4\sin t, 0 \rangle \cdot \langle -5\sin t, 4\cos t, 0 \rangle.$$

$$= -2\big[(5\cos t+3)(-5\sin t) + (4\sin t)(4\cos t)\big] \Big/ \big[(5\cos t+3)^2 + (4\sin t)^2\big]$$

<u>Directional derivatives</u>

$f : \mathbb{R}^2 \to \mathbb{R}$

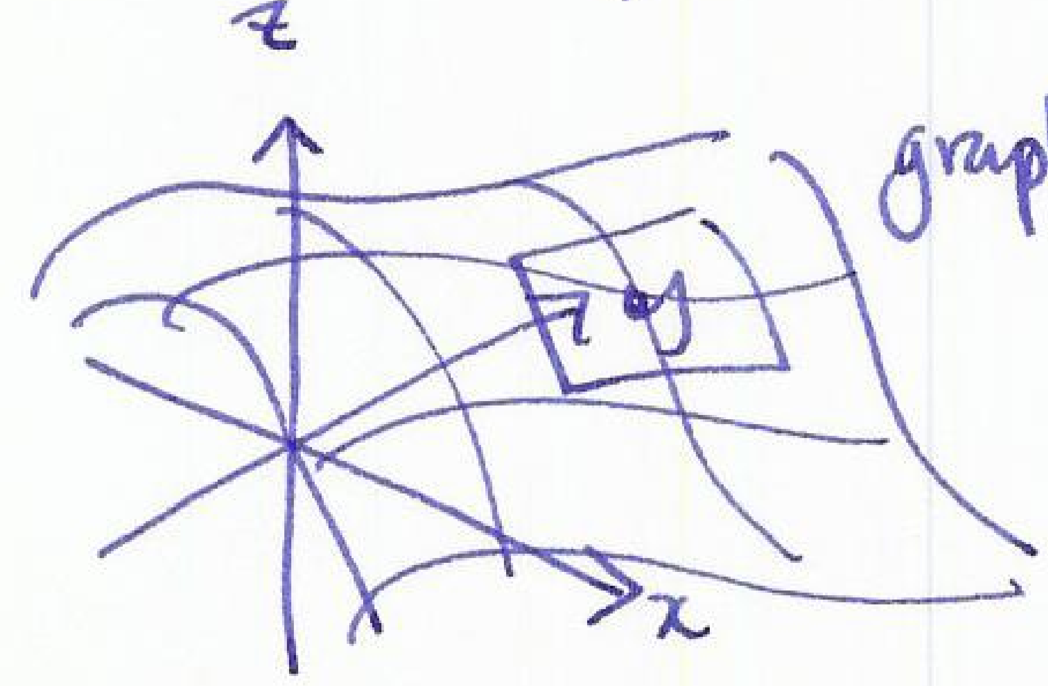
$f(x,y)$



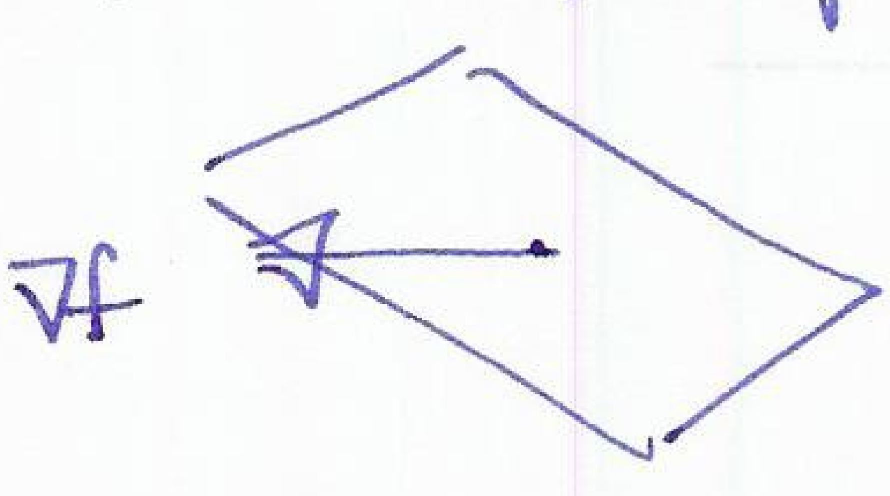
level sets $\nabla f \perp$ level sets

rate of change in direction $\nabla f = \|\nabla f\|$

rate of change in $\perp$ direction tangent to level set is 0.



graph $z = f(x,y)$ tangent plane at a point.

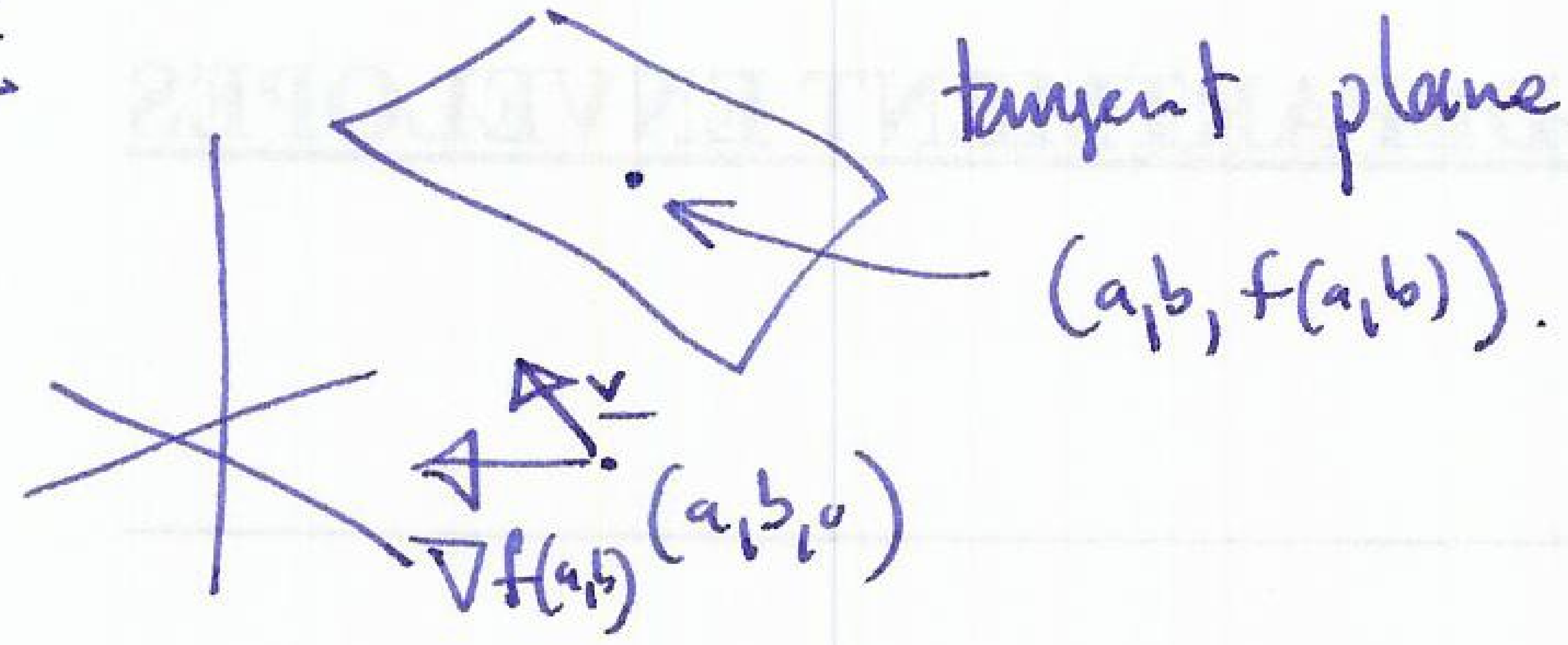


what about rate of change in some other direction $\underline{v}$ (unit vector $\|\underline{v}\|=1$)

<u>Thm</u> the <u>directional derivative</u> $D_{\underline{v}} f$ is equal to

$$\boxed{D_{\underline{v}} f(a,b) = \nabla f(a,b) \cdot \underline{v}}$$

<u>sketch:</u>



tangent plane has equation

$$z = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

so if $\underline{v} = \langle v_1, v_2 \rangle$ then rate of change is $\frac{\partial f}{\partial x}(a,b) \cdot v_1 + \frac{\partial f}{\partial y}(a,b) v_2$

$= \nabla f \cdot \underline{v}$.

<u>Useful properties</u>

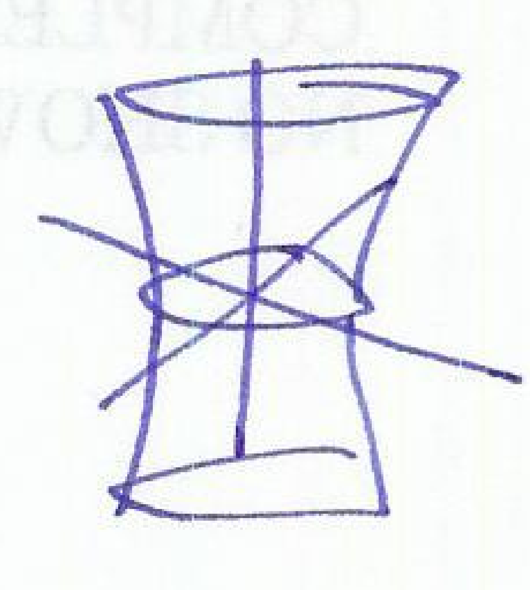
- if $\underline{v}$ is not a unit vector define $D_{\underline{v}} f(a,b) = \nabla f \cdot \underline{v}$
- $D_{\lambda \underline{v}} f(a,b) = \lambda \nabla f \cdot \underline{v}$
- so if $\underline{v}$ is <u>not</u> a unit vector the directional derivative in direction $\underline{v}$ is $\frac{1}{\|\underline{v}\|} \nabla f \cdot \underline{v}$

<u>Applications</u>

<u>Finding normal vectors</u>: sphere $x^2+y^2+z^2=r^2$

consider $f(x,y,z) = x^2+y^2+z^2$

then $\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$ is normal vector.



<u>Finding a tangent plane</u>: hyperboloid $x^2+y^2 = z^2+1$

find normal vectors: consider $f(x,y,z) = x^2+y^2-z^2 = r$

then $\nabla f = \langle 2x, 2y, -2z \rangle$

so normal vector at $(1,1,1)$ is $\langle 2,2,-2 \rangle$

so tangent plane is $\underline{n} \cdot (\underline{x} - \underline{p}) = 0$

$$\langle 2,2,-2\rangle \cdot (\langle x,y,z\rangle - \langle 1,1,1\rangle) = 0$$

$$2x + 2y - 2z = 2.$$

§14.6 Chain rule

$f: \mathbb{R} \to \mathbb{R}^2$

$t \mapsto (x(t), y(t))$

$f'(t) = \langle x'(t), y'(t) \rangle$.

$f: \mathbb{R}^2 \to \mathbb{R}$.

$(x,y) \mapsto f(x,y)$

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

recall: $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$\mathbb{R}^a \xrightarrow{f} \mathbb{R}^b$ looks like $f(x_1, x_2, \ldots, x_a) = (f_1(x_1, \ldots, x_a), f_2(x_1, \ldots, x_a), \ldots f_b(x_1, \ldots x_a))$

$(x_1, x_2, \ldots x_a)$

Example $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = (x^2+y^2, x^2-y^2)$

the derivative at a point is a linear map $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

$(x,y) \mapsto (ax+by, cx+dy)$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}.$$

General chain rule: $\quad \mathbb{R}^a \xrightarrow{g} \mathbb{R}^b \xrightarrow{f} \mathbb{R}^c$

$\underline{x}$

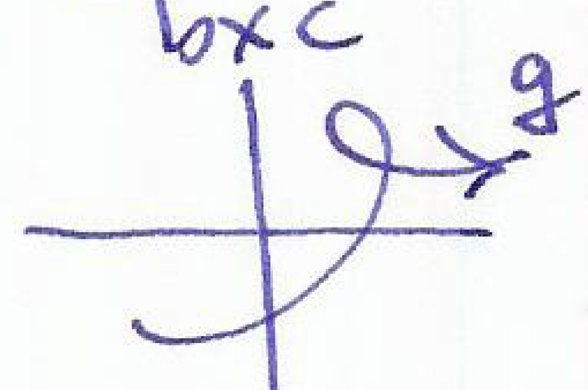
$D(f \circ g)(\underline{x}) = Df(g(\underline{x})) \cdot Dg(\underline{x})$ matrix multiplication!

$$\left[\frac{\partial f_i}{\partial x_j} \right] \left[\frac{\partial g_j}{\partial x_k} \right] \text{ is an } a \times c \text{ matrix!}$$

$\qquad a \times b \qquad\quad b \times c$

Example ① $\mathbb{R} \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$.

$\quad t \qquad\; (x,y)$



$Dg = g'(t) \quad Df = \nabla f$

$g(t) = (g_1(t), g_2(t)) \qquad \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$g'(t) = (g_1'(t), g_2'(t))$

$$D(f(g(x))) = Df(g(t)) \cdot Dg(t).$$

$$= \nabla f \cdot g'(t) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dg_1}{dt}, \frac{dg_2}{dt} \right\rangle$$

$$= \frac{\partial f}{\partial x}\frac{dg_1}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

② $\underset{x_1,x_2}{\mathbb{R}^2} \xrightarrow{g} \underset{y_1,y_2}{\mathbb{R}^2} \xrightarrow{f} \mathbb{R}^2$

$$g(x_1,x_2) = (g_1(x_1,x_2), g_2(x_1,x_2)).$$

$$D(f(g(\underline{x}))) = Df(g(\underline{x})) \cdot Dg(\underline{x}).$$

$$Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}.$$

$$f(y_1,y_2) = f_1(y_1,y_2), f_2(y_1,y_2)$$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}.$$

$$Df(g(\underline{x})).Dg = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2} \cdot \frac{\partial g_2}{\partial x_2} & \cdots \\ & \cdots \end{bmatrix}$$