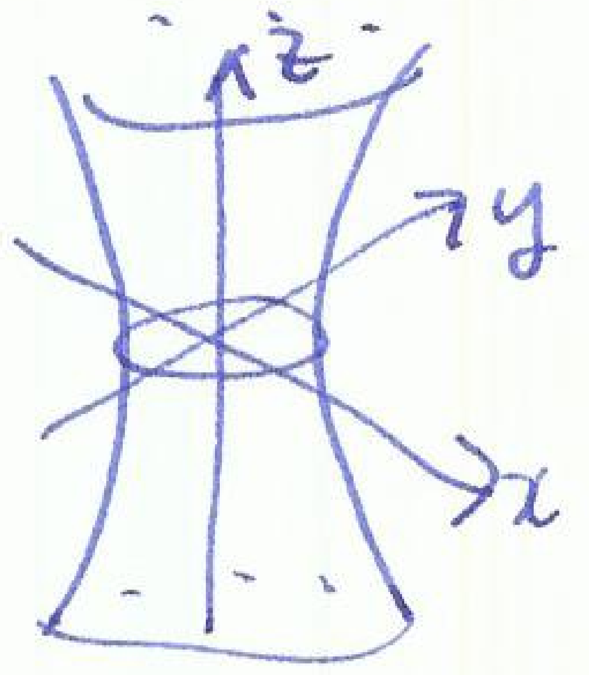
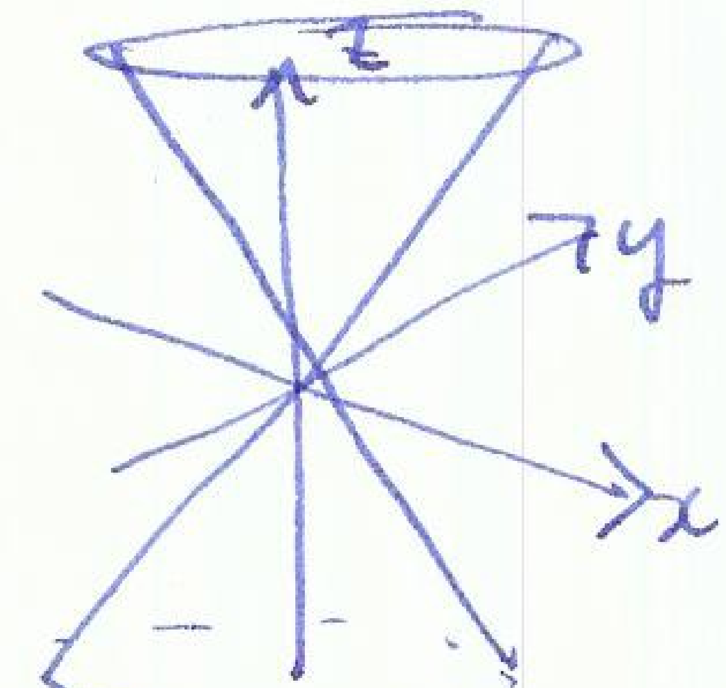
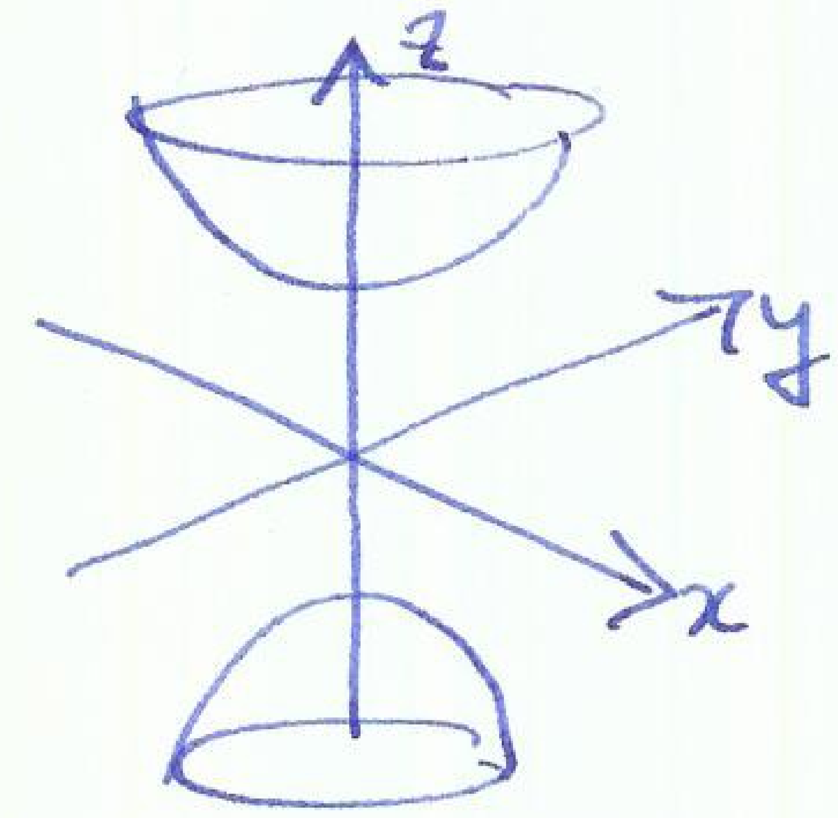
(2) $f(x,y,z) = x^2+y^2-z^2$



$f(x,y,z) = c > 0$



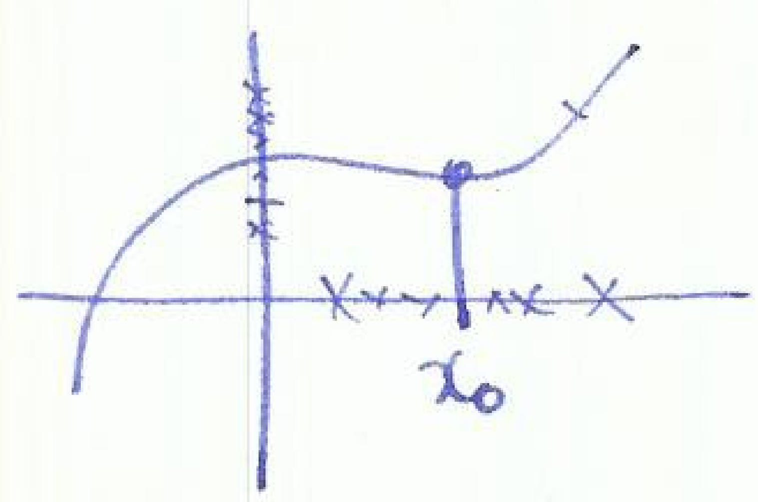
$f(x,y,z) = 0$



$f(x,y,z) = c \; (c<0)$.

§14.2 Limits and continuity for functions of many variables

<u>recall</u>: $y = f(x)$

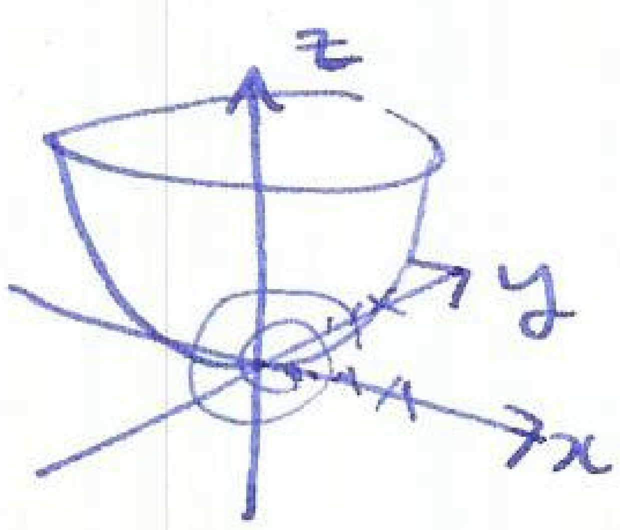


$\lim\limits_{x \to x_0} f(x) = L$

if $|f(x) - L|$ gets small as $|x_0 - x|$ gets small.

only two directions to get to x_0: left or right.

<u>2 vars</u> $z = f(x,y)$

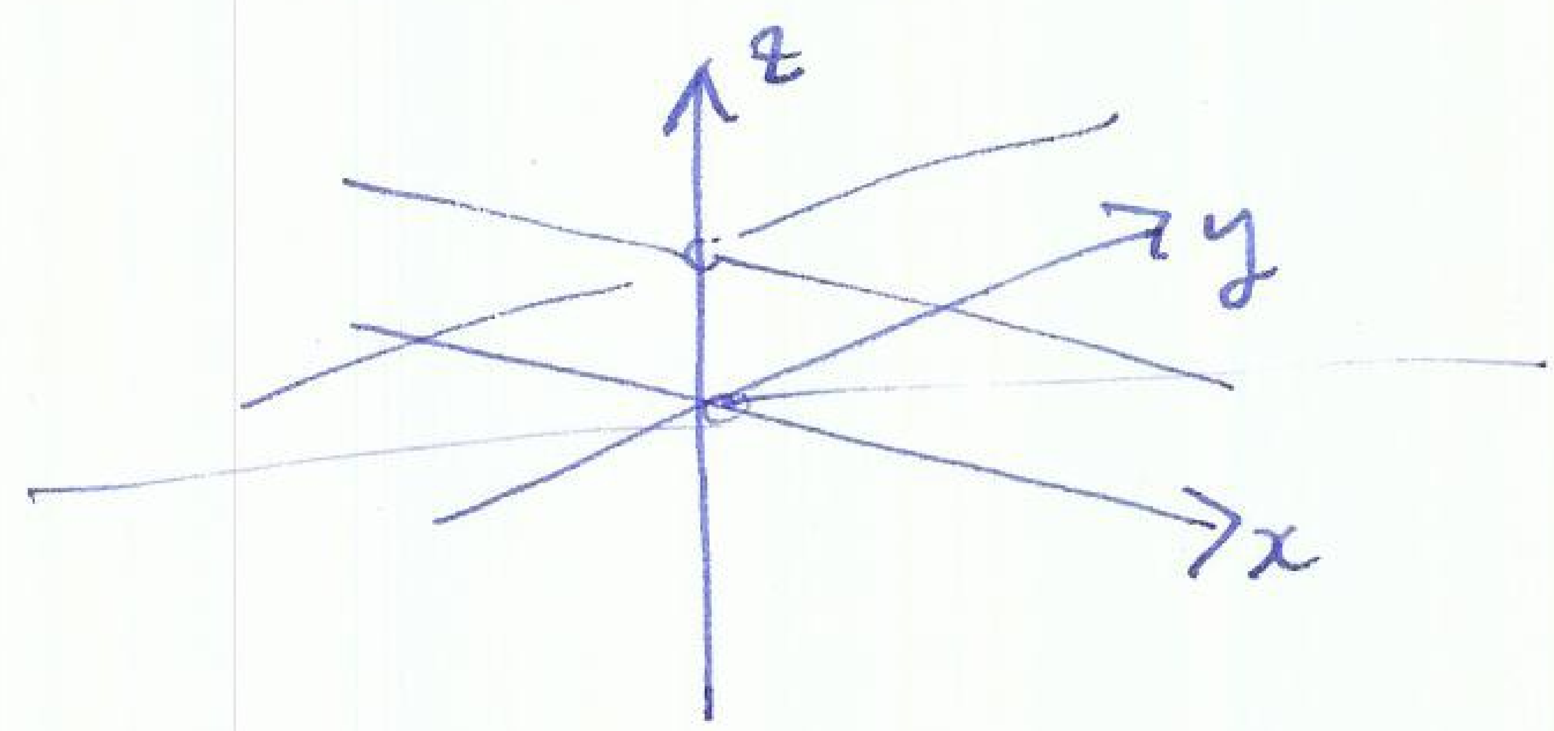


<u>many</u> ways to get to $(0,0)$.

<u>Defn</u> $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $|f(x,y) - L|$ gets small as $|(x,y) - (x_0,y_0)|$ gets small.

<u>Bad example</u> $f(x,y) = \left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2$

<u>Q</u>: what happens near $(0,0)$?



$\lim\limits_{(x,0)\to(0,0)} f(x,y) = \left(\dfrac{x^2}{x^2}\right)^2 = 1$

$\lim\limits_{(0,y)\to(0,0)} f(x,y) = \left(\dfrac{-y^2}{y^2}\right)^2 = 1$ but $\lim\limits_{(x,x)\to(0,0)} = \left(\dfrac{0}{2x^2}\right)^2 = 0$

trick : use polars $x = r\cos\theta$, $y = r\sin\theta$:

$$\left(\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\cos^2\theta}\right)^2 = (\cos^2\theta - \sin^2\theta) = \cos^2 2\theta .$$

Warning : $\lim\limits_{(x,0)\to(0,0)} f(x,y)$ exists $\not\Rightarrow$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists.

Thm Limit laws. (assume $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and $\lim\limits_{(x,y)\to(a,b)} g(x,y)$ exists)

then

sums: $\lim\limits_{(x,y)\to(a,b)} f(x,y)+g(x,y) = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y)$

constant multiple : $\lim\limits_{(x,y)\to(a,b)} kf(x,y) = k\lim\limits_{(x,y)\to(a,b)} f(x,y)$

products : $\lim\limits_{(x,y)\to(a,b)} f(x,y)g(x,y) = \lim\limits_{(x,y)\to(a,b)} f(x,y) \lim\limits_{(x,y)\to(a,b)} g(x,y)$

quotient : $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim\limits_{(x,y)\to(a,b)} f(x,y)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)}$ as long as $\lim\limits_{(x,y)\to(a,b)} g(x,y) \neq 0$.

Defn Continuity $f(x,y)$ is continuous at (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Q: when is $f(x,y)$ cts? A: hard to know.

However : Thm "compositions of continuous functions are cts".

so if f, g cts then fg cts

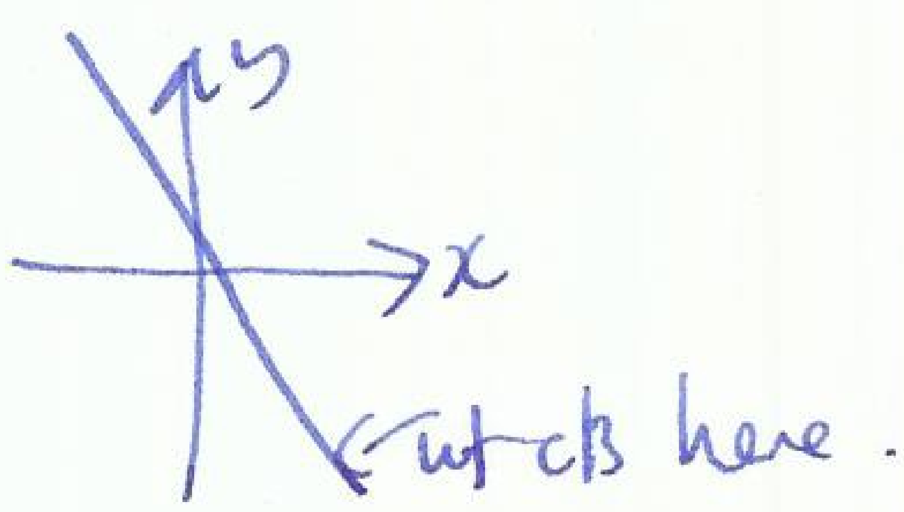
$f \pm g$ cts

f/g cts. (if $g \neq 0$)

$f \circ g$ cts.

<u>Example</u> $f(x,y) = \dfrac{x-y}{x+y}$ note $\left.\begin{array}{l} f(x,y) = x \text{ cts.} \\ f(x,y) = y \text{ cts} \end{array}\right\}$ linear functions are cts.

so $x-y$ cts $x+y$ cts. therefor $\dfrac{x-y}{x+y}$ is cts as long as $x+y \neq 0$ (i.e. $x \neq -y$).



How to show a limit does not exist: $f(x,y) = \dfrac{x^2}{x^2+y^2}$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE.

just find some directions to get to $(0,0)$ which give different answers.

$$\left.\begin{array}{l} \lim\limits_{(x,0)\to(0,0)} \dfrac{x^2}{x^2+y^2} = \lim\limits_{(x,0)\to(0,0)} 1 = 1 \\ \lim\limits_{(0,y)\to(0,0)} \dfrac{x^2}{x^2+y^2} = \lim\limits_{(0,y)\to(0,0)} 0 = 0 \end{array}\right\} \Rightarrow \text{DNE}.$$

similar Def^{n}s for functions of 3 vars.

<u>Def^{n}</u> $f(x,y,z)$ cts at (x_0, y_0, z_0) if $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0)$

<u>Th^{m}</u> "compositions of cts functions are cts".

so $f(x,y,z) = \dfrac{1}{x^2+y-z}$ cts when $x^2+y-z \neq 0$.

§14.3 Partial derivatives

$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y)$

$\frac{\partial f}{\partial x}(x,y) = f_x =$ "differentiate wrt x keeping y constant"

$\frac{\partial f}{\partial y}(x,y) = f_y =$ "differentiate wrt y keeping x constant"

<u>Example</u> $f(x,y) = x^2 + y^2 \qquad f_x = 2x \qquad f_y = 2y$

$f(x,y) = xy \qquad f_x = y \qquad f_y = x$

$f(x,y) = xye^x + x^2y \qquad f_x = ye^x + xye^x + 2xy$

$f_y = xe^x + x^2$

<u>note</u> f_x, f_y are still functions of two variables, so we can partially differentiate again.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{xx} = \frac{\partial^2 \cancel{f}}{\partial x^2}.$$

<u>Example</u>

$$f_{xx} = ye^x + xye^x + ye^x + 2y$$

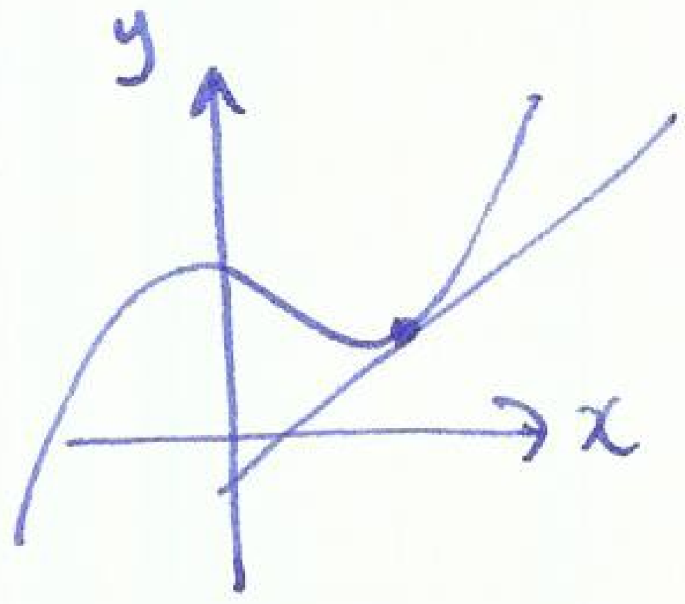
$$\left.\begin{aligned} f_{xy} &= e^x + xe^x + 2x \\ f_{yx} &= xe^x + e^x + 2x \end{aligned}\right\} \text{equal!}$$

$$f_{yy} = 0$$

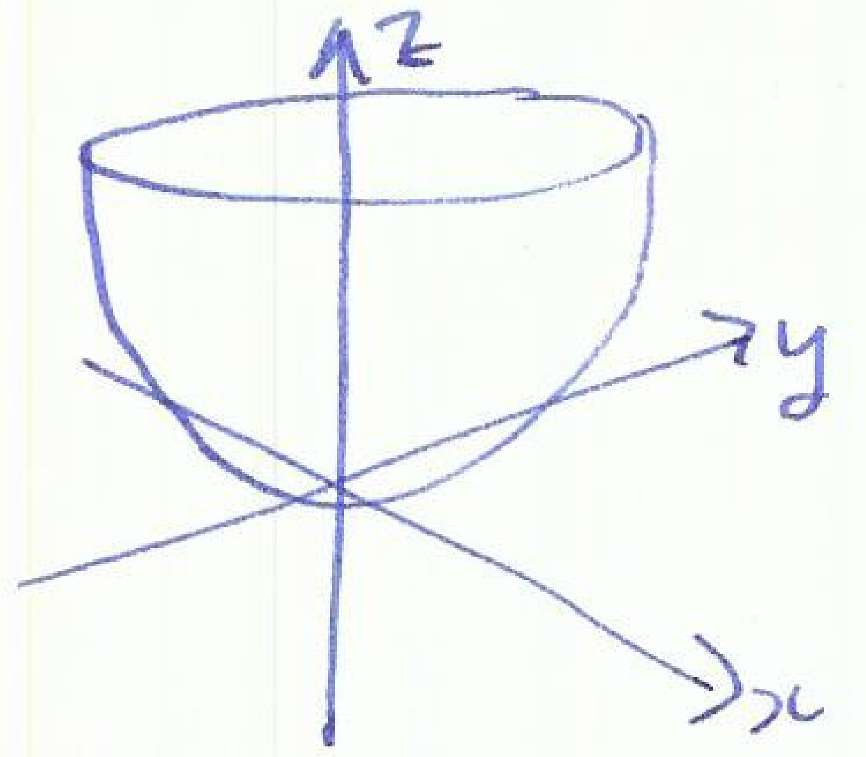
<u>Thm</u> If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$

"mixed partials are equal".

what does this mean?

1d 

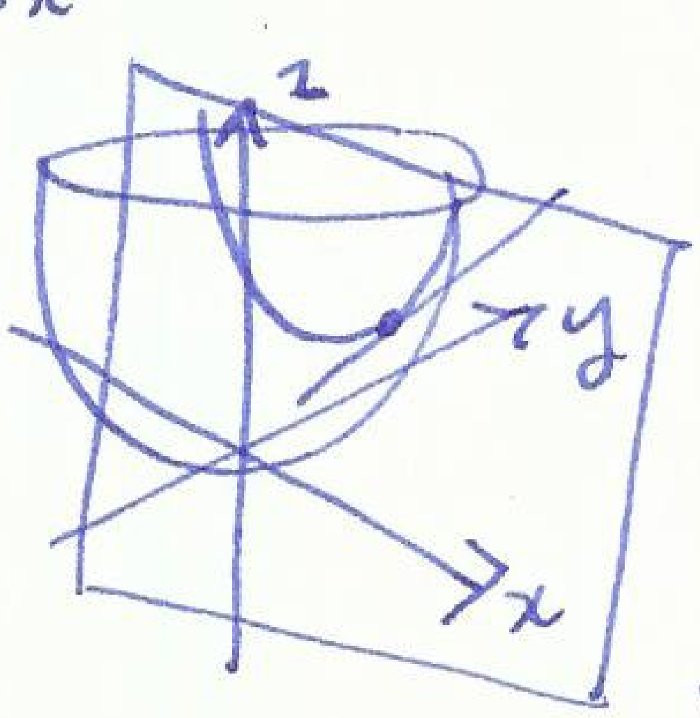
$f: \mathbb{R} \to \mathbb{R}$ $\quad \dfrac{\partial f}{\partial x} = \dfrac{df}{dx}$ = rate of change | slope of tangent line

2d $f(x,y) = x^2 + y^2$ 

$f_x(x,y) = 2x$

$f_y(x,y) = 2y$

f_x = diff$^{\underline{n}}$ wrt x keeping y fixed, i.e. $y = c \Leftrightarrow$ vertical slice parallel to xz-plane

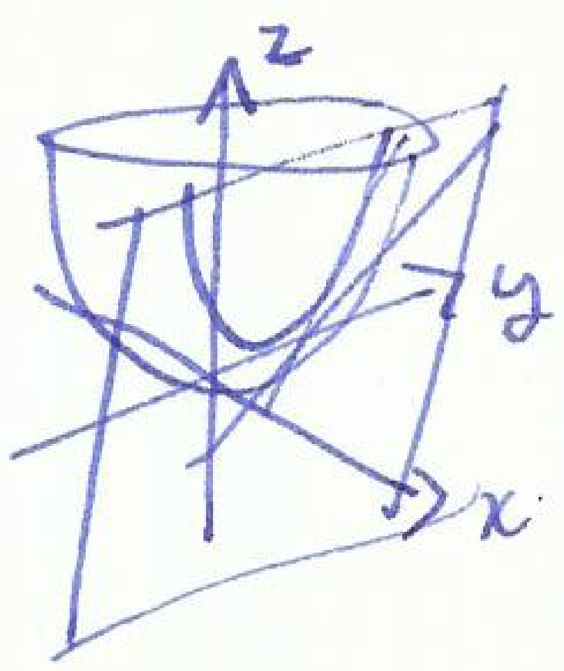


in $y=c$ f looks like $f(x,c) = x^2 + c^2$

(i.e. a parabola with slope $2x$).

$$\boxed{\frac{\partial f}{\partial x} = \text{slope in } x\text{-direction.}}$$

keep x fixed: i.e. $x = c \Leftrightarrow$ vertical slice parallel to yz-plane



in $x=c$ f looks like $f(y) = c^2 + y^2$

(i.e. a parabola with slope $2y$).

$$\boxed{\frac{\partial f}{\partial y} = \text{slope in } y\text{-direction}}$$

f_{xx} = "2nd derivative in x-direction"

f_{yy} = "2nd derivative in y-direction"

f_{xy} = "rate of change of f_x in y-direction"

f_{yx} = "rate of change of f_y in x-direction".