## §13.5 Motion in $\mathbb{R}^3$

$\underline{r}(t)$

Location / position $\underline{r}(t)$

velocity $\underline{v}(t) = \underline{r}'(t) = \lim_{h \to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}$

speed $\| \underline{v}(t) \|$

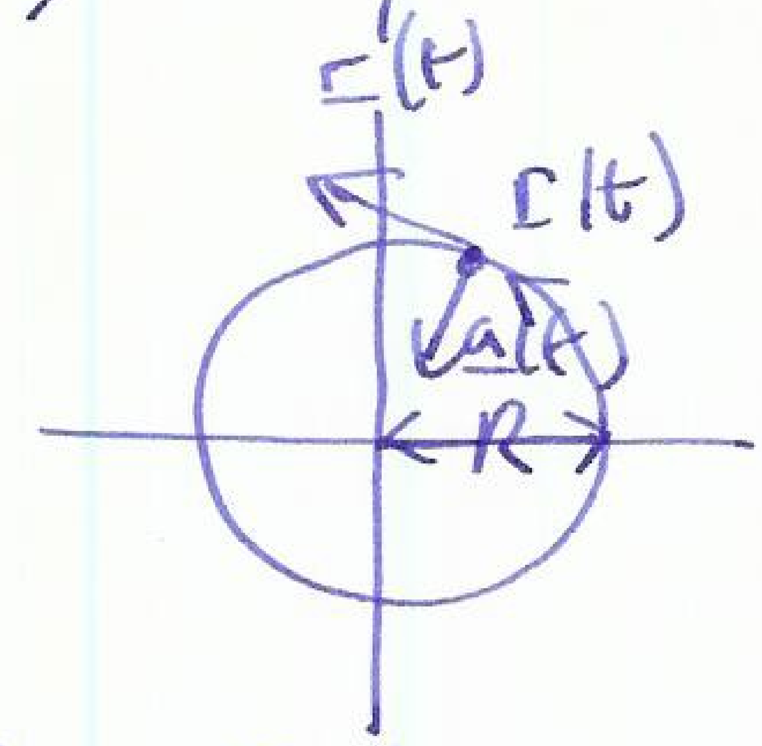
acceleration $\underline{a}(t) = \underline{r}''(t)$

<u>Example</u> suppose position given by $\underline{r}(t) = \langle e^{-t}, \ln t, \sqrt{t} \rangle$.

then velocity $\underline{v}(t) = \underline{r}'(t) = \langle -e^{-t}, \frac{1}{t}, \frac{1}{2} t^{-1/2} \rangle$

acceleration $\underline{a}(t) = \underline{r}''(t) = \langle e^{-t}, -t^{-2}, -\frac{1}{4} t^{-3/2} \rangle$

<u>Example</u> Acceleration for uniform circular motion




$\underline{r}(t) = \langle R\cos\omega t, R\sin\omega t \rangle$ $\omega$ constant called <u>angular velocity</u>

if speed is constant $v$, then $\| r'(t) \| = \| \langle -R\omega\sin\omega t, \omega R\cos\omega t \rangle \|$

$= \sqrt{R^2\omega^2 (\sin^2\omega t + \sin^2\omega t)} = R\omega$.

so $v = R\omega \iff \omega = \frac{v}{R}$

acceleration $\underline{a}''(t) = \langle -R\omega^2 \cos\omega t, -R\omega^2 \cos\omega t \rangle$

so $\| \underline{a}''(t) \| = R\omega^2 = \frac{R v^2}{R^2} = \frac{v^2}{R}$

so the acceleration vector $\underline{a}(t)$ has constant length $\frac{v^2}{R}$ and points towards the origin

<u>Observation</u> we can go backwards, i.e. if we know acceleration $\underline{a}(t)$

then $\underline{v}(t) = \int_0^t \underline{a}(u)\,du + \underline{v}_0$ $\underline{v}_0$ = initial velocity at $t=0$

$\underline{r}(t) = \int_0^t \underline{v}(u)\,du + \underline{r}_0$ $\underline{r}_0$ = initial position at $t=0$

<u>Example</u> find $\underline{r}(t)$ if $\underline{a}(t) = \langle 1, t \rangle$, $\underline{v}_0 = \langle 1,1 \rangle$
$\underline{r}_0 = \langle 2,3 \rangle$

$$\underline{v}(t) = \int_0^t \langle 1, u \rangle \, du + \underline{v}_0$$

$$= \left\langle \int_0^t du, \int_0^t u \, du \right\rangle + \langle 1,1 \rangle$$

$$= \left\langle [u]_0^t, \left[\tfrac{1}{2}u^2\right]_0^t \right\rangle + \langle 1,1 \rangle$$

$$= \langle t, \tfrac{1}{2}t^2 \rangle + \langle 1,1 \rangle = \langle t+1, \tfrac{1}{2}t^2+1 \rangle$$

$$\underline{r}(t) = \int_0^t \underline{v}(u) \, du + \underline{r}_0$$

$$= \int_0^t \langle u+1, \tfrac{1}{2}u^2+1 \rangle \, du + \langle 2,3 \rangle$$

$$= \left[ \langle \tfrac{1}{2}u^2 + u, \tfrac{1}{6}u^3 + u \rangle \right]_0^t + \langle 2,3 \rangle$$

$$= \langle \tfrac{1}{2}t^2 + t, \tfrac{1}{6}t^3 + t \rangle + \langle 2,3 \rangle = \langle \tfrac{1}{2}t^2+t+2, \tfrac{1}{6}t^3+t+3 \rangle.$$

<u>Newton's Law of motion</u>

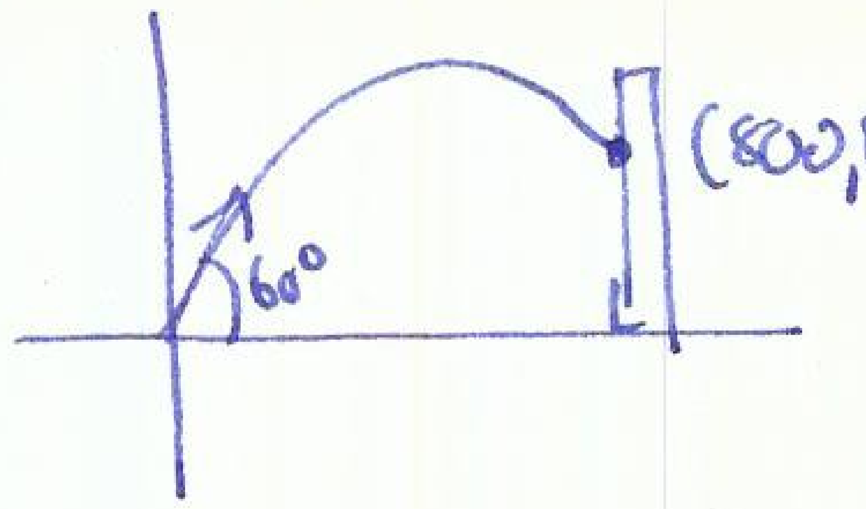
$F = ma$ vector form $\underline{F} = m\underline{a}$

<u>note</u> if object has position $\underline{r}(t)$, this works if $\underline{F}$ not constant,

i.e. $\underline{F}$ can depend on position: $\underline{F}(\underline{r}(t))$

then $\underline{F}(\underline{r}(t)) = m\underline{a}(t) = m \cancel{a} \underline{r}''(t)$.

<u>Example</u>




an object is thrown at angle of 60°. what initial speed do you need to hit a point 500ft up 800ft away

gravity: downward force of magnitude $mg$ $\quad g = 32\,\text{ft}/\text{sec}$.

in vector form $\underline{F} = \langle 0, -gm \rangle$

$$\underline{F} = m\underline{r}''(t) \Rightarrow \underline{r}''(t) = \langle 0, -g \rangle$$

integrate wrt $t$: $\quad \underline{r}'(t) = \int_0^t \underline{r}''(u)\,du = \int_0^t \langle 0, -32 \rangle\,du = \langle 0, -32t \rangle + \underline{v_0}$

integrate wrt $t$: $\quad \underline{r}(t) = \int_0^t \underline{r}'(u)\,du = \int_0^t \langle 0, -32u \rangle + \underline{v_0}\,du$

$$= \Big[\langle 0, -16u^2 \rangle + \underline{v_0}\,u\Big]_0^t + \underline{r_0}$$

$$= \langle 0, 16t^2 \rangle + \underline{v_0}\,t + \underline{r_0}$$

$\underline{r_0} = \underline{0} \qquad \underline{v_0} = v_0 \langle \cos 60°, \sin 60° \rangle = v_0 \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

so $\quad \underline{r}(t) = \langle 0, -16t^2 \rangle + t v_0 \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

now solve for $v_0$:

$$\underline{r}(t) = \langle 800, 500 \rangle = \langle 0, -16t^2 \rangle + t v_0 \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle.$$

$800 = \frac{1}{2} t v_0 \qquad 500 = -16t^2 + \frac{\sqrt{3}}{2} t v_0.$

$\Downarrow$

$t = \dfrac{1600}{v_0}$

$$\Rightarrow \quad 500 = -16\left(\frac{1600}{v_0}\right)^2 + \frac{\sqrt{3}}{2}\,\frac{1600}{\cancel{v_0}}\,\cancel{v_0}.$$

$$v_0^2 = \frac{16 \cdot (1600)^2}{800\sqrt{3} - 500} \approx 215\,\text{ft}/\text{s}.$$