Fundamental theorem of calculus (vector-valued version)

$\underline{r}(t)$ continuous on $[a,b]$ and $\underline{R}(t)$ antiderivative of $\underline{r}(t)$. Then

$$\int_a^b \underline{r}(t)\,dt = \underline{R}(b) - \underline{R}(a).$$

Example A particle moves with velocity $\underline{r}(t) = \langle t, \sin t\rangle$

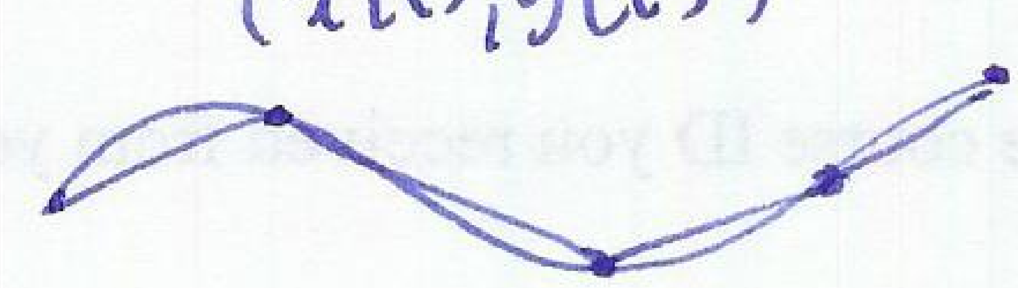
if it starts at $\langle 1, 1\rangle$ where is it at time 4?

$$\underline{R}(t) = \langle \tfrac{1}{2}t^2, -\cos t\rangle + \underline{c}$$

$$\int_0^4 \underline{r}(t)\,dt = \underline{R}(4) - \underline{R}(0)$$

$$= \langle 8, -\cos 4\rangle - \langle 0, 1\rangle = \langle 8, 1-\cos 4\rangle.$$

§13.3 Arc length and speed



Recall: arc length of plane curves:

parameterized curve $(x(t), y(t))$ with $t \in [a,b]$

$$\text{length} \quad L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

this generalizes to parameterized curves in $\mathbb{R}^3$: $\underline{r}(t) = (x(t), y(t), z(t))$ $t \in [a,b]$

$$\text{length} \quad L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_a^b \|\underline{r}'(t)\|\,dt.$$

Example Find the arc length of $\underline{r}(t) = \langle \cos 2t, \sin 2t, 2t\rangle$ for $t \in [0, 2\pi]$.

$$\underline{r}'(t) = \langle -2\sin 2t, +2\sin 2t, 2\rangle$$

$$\|\underline{r}'(t)\| = \sqrt{4\sin^2 2t + 4\sin^2 2t + 4} = \sqrt{4+4} = 2\sqrt{2}$$

$$L = \int_a^b \|\underline{r}'(t)\|\,dt = \int_0^{2\pi} 2\sqrt{2}\,dt = 2\sqrt{2}\,[t]_0^{2\pi} = 4\pi\sqrt{2}.$$

<u>Observation</u> $\underline{r}'(t)$ is tangent to the curve (as long as $\underline{r}'(t) \neq \underline{0}$)

$\|\underline{r}'(t)\|$ is the <u>speed</u> at time t.

<u>Example</u> If a particle moves with position given by $\underline{r}(t) = \langle e^{2t}, t^{1/3}, \tan t \rangle$ find the speed at $t = 2$

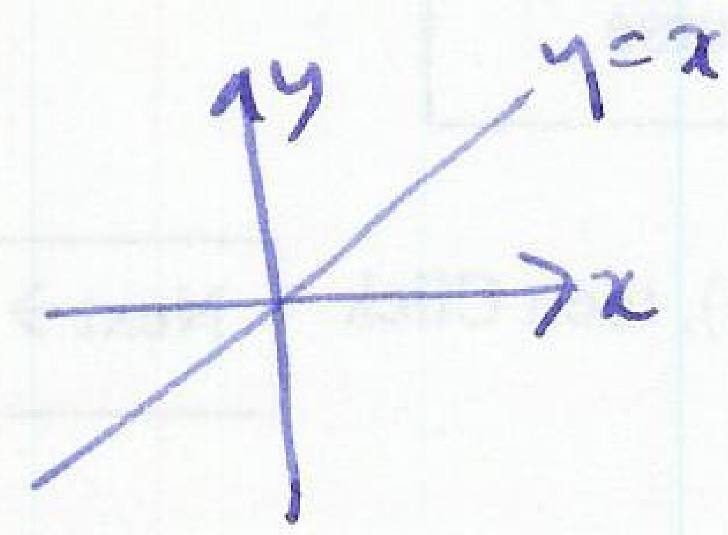
$$\underline{r}'(t) = \langle 2e^{2t}, \tfrac{1}{3}t^{-2/3}, \sec^2 t \rangle$$

$$\|\underline{r}'(2)\| = \left\| \left\langle 2e^4, \frac{1}{3 \cdot 2^{2/3}}, \sec^2(2) \right\rangle \right\| = \sqrt{4e^8 + \frac{1}{9 \cdot 2^{4/3}} + \sec^4(2)}.$$

<u>Arc length parameterizations</u>

<u>Problem</u>: parameterizations are not unique.

<u>example</u>



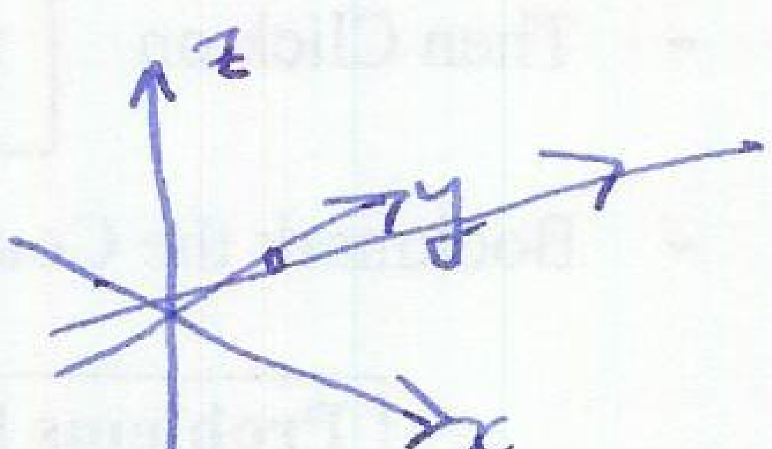
$$\left.\begin{array}{l} \underline{r}(t) = \langle t, t \rangle \\ \underline{r}(t) = \langle t^3, t^3 \rangle \end{array}\right\} \text{ both parameterize } y = x.$$

Special <u>parameterizations</u>: <u>arc length</u> or <u>unit speed</u> parameterization

<u>Def</u> $\underline{r}(t)$ is an arc length parameterization if $\|\underline{r}'(t)\| = 1$ for all t.

(i.e. move along the curve at unit speed)

<u>Example</u> find an arc length parameterization for $\underline{r}(t) = \langle 2t, 1-2t, t \rangle$



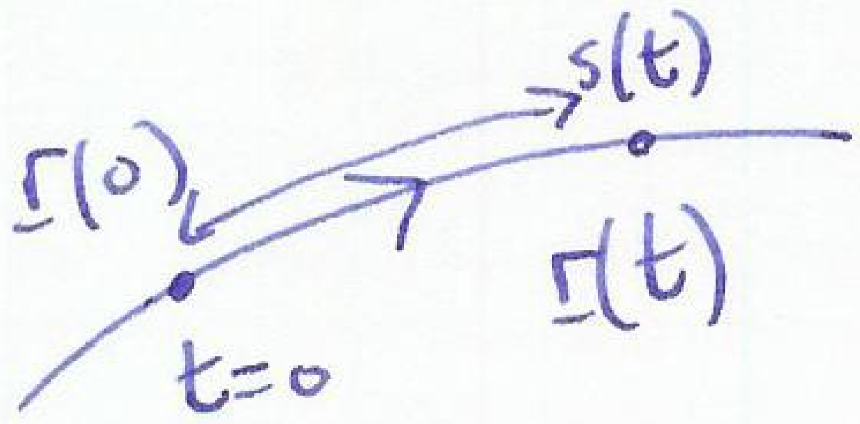
① find the arc length at time t

arc length at time t $\quad s(t) = \int_0^t \|\underline{r}'(u)\|\, du$

$$\|\underline{r}'(u)\| = \|\langle 2, -2, 1 \rangle\| = 3.$$

$$s(t) = \int_0^t 3\, du = [3u]_0^t = 3t$$

② find the inverse function for arc length



instead of $\underline{r}(t)$ want $\underline{r}(s^{-1}(t))$

want: $\underline{r}(4)$ to be $\underline{r}(t)$ when $s(t)=4$

$t = s^{-1}(4)$

so $\underline{r}(4) = \underline{r}(s^{-1}(4))$

here $s(t) = 3t$ so $s^{-1}(t) = t/3$.

③ write down reparameterized curve:

$$\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t)) = \underline{r}\left(\frac{t}{3}\right) = \left\langle 2\cdot\frac{t}{3}, 1 - 2\frac{t}{3}, \frac{t}{3} \right\rangle.$$

check $\|\underline{r}'(t)\| = \left\| \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle \right\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1.$

Summary $\underline{r}(t)$ arbitrary parameterization

let $s(t)$ be arc length from start on $[0,t]$.

i.e. $s(t) = \int_0^t \|\underline{r}'(u)\|\, du$

and let $s^{-1}(t)$ be the inverse function

then the arc length parameterization is

$$\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t))$$

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check this is really unit speed: i.e. $\|\hat{\underline{r}}'(t)\| = 1$.

$$\hat{\underline{r}}'(t) = \frac{d}{dt}\left(\underline{r}(s^{-1}(t))\right) = \underline{r}'(s^{-1}(t))\, \frac{d}{dt}\left(s^{-1}(t)\right)$$

recall: $\frac{d}{dt}\left(s^{-1}(t)\right) = \frac{1}{s'(s^{-1}(t))}$ why? $s^{-1}(t) = x \iff t = s(x)$

$\frac{d}{dt}(s^{-1}(t)) = \frac{dx}{dt}$ $\qquad \frac{dt}{dt} = 1 = s'(x)\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{1}{s'(x)} = \frac{1}{s'(s^{-1}(t))}$

$$\hat{\underline{r}}'(t) = \underline{r}'(s^{-1}(t)) \cdot \frac{1}{s'(s^{-1}(t))}$$

so $\|\hat{\underline{r}}'(t)\| = \|\underline{r}'(s'(t))\| \cdot \dfrac{1}{\|s'(s'(t))\|}$

recall $s(t) = \int_0^t \|\underline{r}'(u)\|\,du \Rightarrow s'(t) = \|\underline{r}'(u)\|$ (fundamental theorem of calculus)

so $\|\hat{\underline{r}}'(t)\| = \|\underline{r}'(s^{-1}(t))\| \cdot \dfrac{1}{\|\underline{r}'(s^{-1}(t))\|} = 1$

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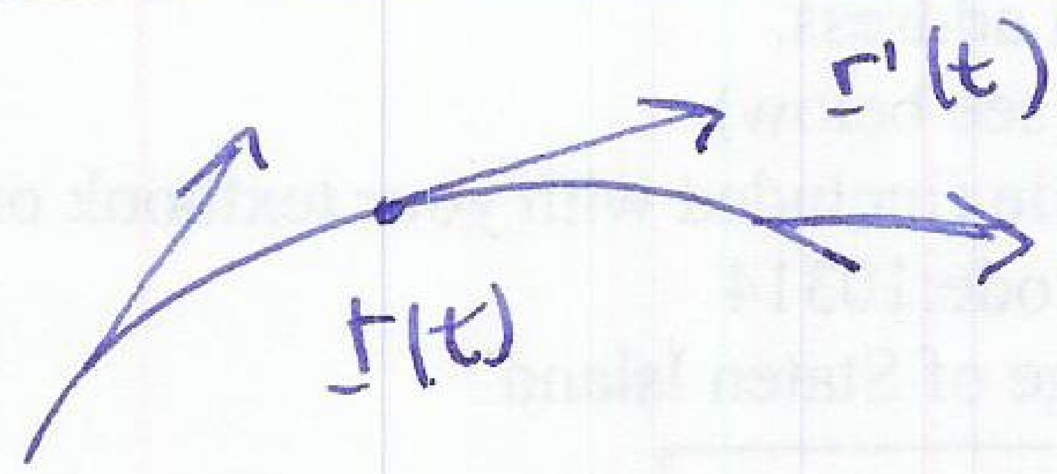
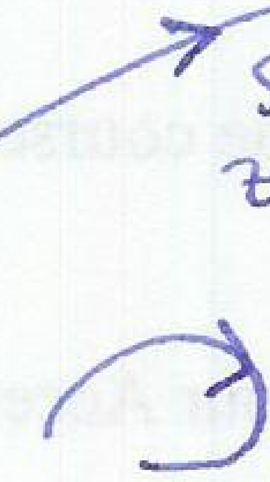
§13.4 Curvature



assumption we have a parameterization where $\underline{r}'(t) \neq \underline{0}$ infinitely.

$\underline{r}'(t)$ points in direction of tangent vector.



Def$^{\underline{n}}$ $T(t) = \dfrac{\underline{r}'(t)}{\|\underline{r}'(t)\|}$ is the unit tangent vector

so $\underline{T}'(t)$ is derivative of unit tangent vector

$\|\underline{T}'(t)\|$ speed of change of unit tangent vector

problem: this depends on parameterization.

solution: use arc length parameterization

Def$^{\underline{n}}$ Curvature $\underline{r}(s)$ arc length parameterization

$\underline{T}(s)$ unit tangent vector

then the curvature $\kappa(s) = \left\| \dfrac{d\underline{T}}{ds} \right\|$

Example <u>straight lines have zero curvature</u>.

$$\underline{r}(t) = \underline{a} + \underline{b}t \quad \text{where } \|\underline{b}\| = 1$$

<u>note</u>: this an arc length parameterization: check

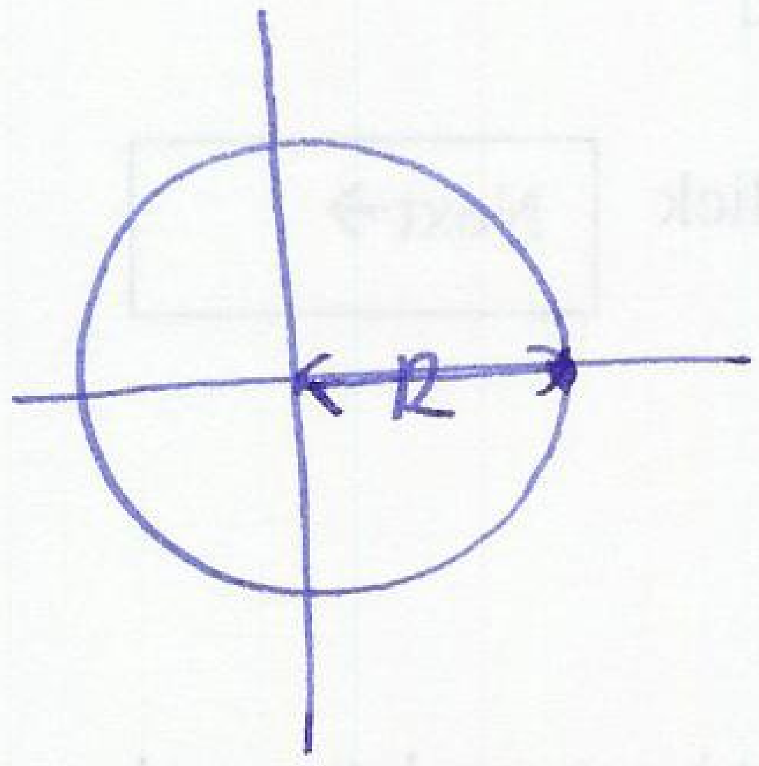
$$\|\underline{r}'(t)\| = \|\underline{b}\| = 1.$$

also: $\underline{r}'(t) = \underline{b}$ so $T(t) = \underline{b}$

so $\frac{d}{dt}\left(T(t)\right) = 0$

so $\kappa(t) = \|\underline{0}\| = 0$.

<u>Example</u> <u>curvature of a circle of radius R is $1/R$</u>



$$\underline{r}(\theta) = \langle R\cos\theta, R\sin\theta \rangle$$

<u>not</u> arc length parameterization

$$\cancel{\underline{r}(t) = \left\langle R\cos\left(\tfrac{t}{R}\right), R\sin\left(\tfrac{t}{R}\right) \right\rangle} \text{ is arclength}$$

find an arc length parameterization

① find arc length $s(t) = \int_0^t \|\underline{r}'(u)\|\,du$

$$= \int_0^t \|\langle -R\sin u, R\cos u \rangle\|\,du$$

$$= \int_0^t R\,du = \left[Ru\right]_0^t = Rt$$

② inverse function $s^{-1}(t) = \frac{t}{R}$

③ arc length parameterization is $\underline{r}(s^{-1}(t)) = \left\langle R\cos\left(\frac{t}{R}\right), R\sin\left(\frac{t}{R}\right) \right\rangle$

unit tangent vector $T(t) = \dfrac{\underline{r}'(t)}{\|\underline{r}'(t)\|} = \left\langle -R\sin\left(\frac{t}{R}\right)\cdot\frac{1}{R}, R\cos\left(\frac{t}{R}\right)\cdot\frac{1}{R} \right\rangle$

$$= \left\langle -\sin\left(\tfrac{t}{R}\right), \cos\left(\tfrac{t}{R}\right) \right\rangle$$

so curvature $\kappa(t) = \left\| \frac{dT}{dt} \right\| = \left\| \langle -\cos\left(\frac{t}{R}\right)\cdot\frac{1}{R}, -\sin\left(\frac{t}{R}\right)\cdot\frac{1}{R} \rangle \right\|$

$= \frac{1}{R}$

Arc length parametrizations are often hard to find.

For a non-arc length parameterization

recall: $\kappa(s) = \left\| \frac{dT}{ds} \right\|$ (arc length parameterization)

for any parameterization let $T(t) = \frac{r'(t)}{\|r'(t)\|}$ and let $s(t)$ be arc length.

$s(t) = \int_0^t \|r'(u)\| du$

then $T'(t) = \frac{dT}{ds}\frac{ds}{dt} = \frac{dT}{ds}\|r'(t)\|$

so $\|T'(t)\| = \left\| \frac{dT}{ds} \right\| \|r'(t)\| = \kappa(t)\|r'(t)\|$

so $\kappa(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$

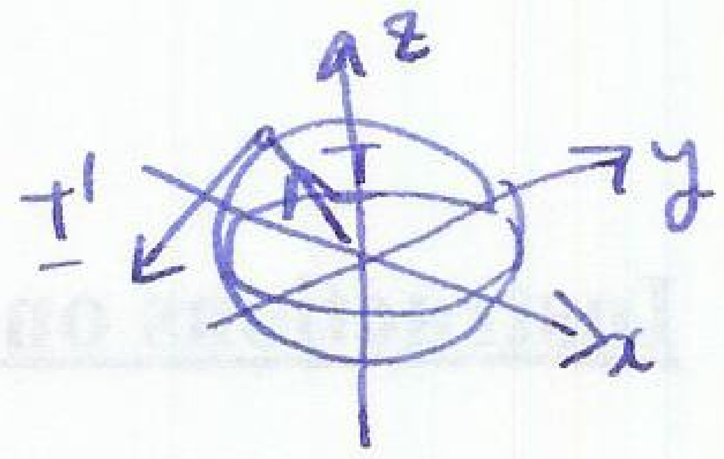
Summary curvature

$\kappa(s) = \left\| \frac{dT}{ds} \right\|$ for arc length parameterizations

$\kappa(s) = \frac{\|T'(t)\|}{\|r'(t)\|}$ for any (regular) parameterization $r'(t) \neq 0$

Thm (curvature in $\mathbb{R}^3$) $\underline{r}(t)$ regular parameterization.

$$\kappa(t) = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{\|\underline{r}'(t)\|^3}$$



Proof recall. $\underline{T}(t)$ and $\underline{T}'(t)$ are orthogonal.

(as $\|\underline{T}(t)\|^2 = 1 \Rightarrow \underline{T}(t) \cdot \underline{T}(t) = 1$.

$$\underline{T}'(t) \cdot \underline{T}(t) + \underline{T}(t) \cdot \underline{T}'(t) = 2\underline{T}(t) \cdot \underline{T}'(t) = 0 \quad)$$

set $v(t) = \|\underline{r}'(t)\|$ then $\|\underline{T}'(t)\| = \kappa(t) v(t)$

so $\|\underline{T}(t) \times \underline{T}'(t)\| = \|\underline{T}(t)\| \, \|\underline{T}'(t)\|$

$$= \|\underline{T}'(t)\|$$

$$= \kappa(t) v(t) \quad \text{so} \quad \kappa(t) = \frac{\|\underline{T}(t) \times \underline{T}'(t)\|}{v(t)}$$

$$\underline{r}'(t) = v(t) \underline{T}(t)$$

so $\underline{r}''(t) = \frac{d}{dt}\left(v(t) T(t)\right) = v'(t)\underline{T}(t) + v(t)\underline{T}'(t)$

$$\underline{r}'(t) \times \underline{r}''(t) = v(t)\underline{T}(t) \times \left(v'(t)\underline{T}(t) + v(t)\underline{T}'(t)\right).$$

$$= v(t)^2 \, \underline{T}(t) \times \underline{T}'(t)$$

so $\|\underline{r}'(t) \times \underline{r}''(t)\| = v(t)^2 \, \|\underline{T}(t) \times \underline{T}'(t)\|$

so $$\kappa(t) = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{(v(t))^3} \quad \square$$

<u>Example</u> find curvature of $\underline{r}(t) = \langle t, t^2, t^3 \rangle$.

$\underline{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

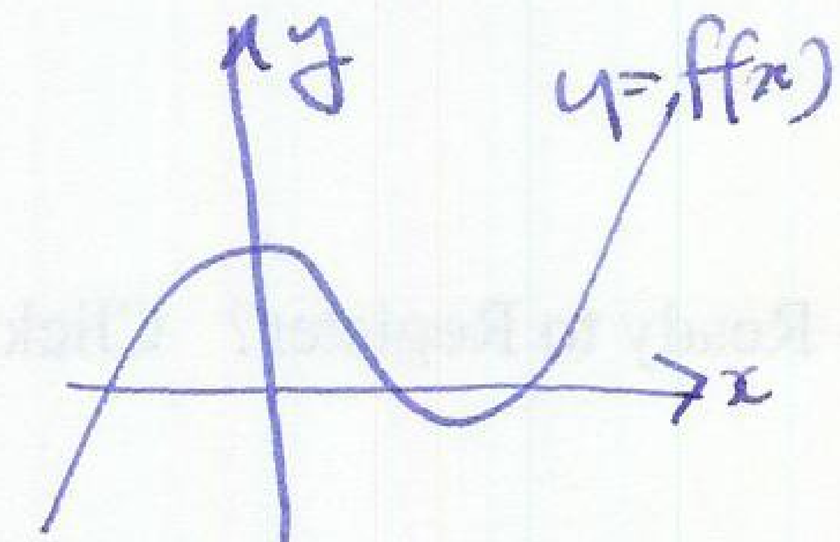
$\underline{r}''(t) = \langle 0, 2, 6t \rangle$

$$\underline{r}'(t) \times \underline{r}''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} \langle 12t^2 - 6t^2, -6t, 2 \rangle$$

$$= \langle 6t^2, -6t, 2 \rangle$$

so $$\kappa(t) = \frac{\| \underline{r}'(t) \times \underline{r}''(t) \|}{\| \underline{r}'(t) \|^3} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}.$$

<u>Thm</u> <u>Curvature of graphs in $\mathbb{R}^2$</u>



$\underline{r}(x) = \langle x, f(x), 0 \rangle$

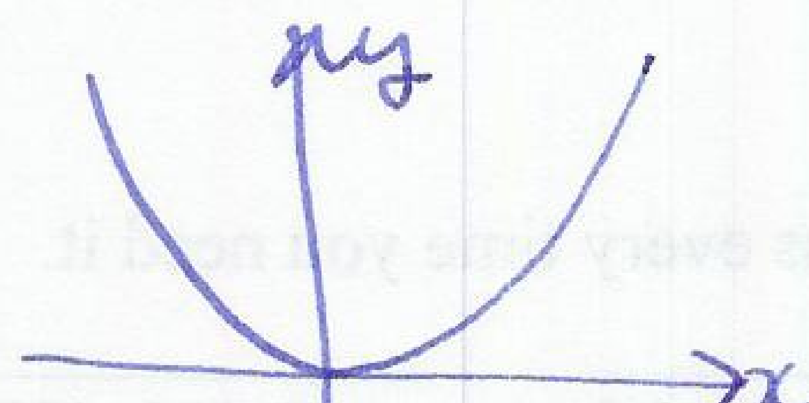
$\underline{r}'(x) = \langle 1, f'(x), 0 \rangle$

$\underline{r}''(x) = \langle 0, f''(x), 0 \rangle$

$$\underline{r}'(x) \times \underline{r}''(x) = \begin{vmatrix} i & j & k \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \langle 0, 0, f''(x) \rangle$$

so $$\kappa(x) = \frac{\| \underline{r}'(x) \times \underline{r}''(x) \|}{\| \underline{r}'(x) \|^3} = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

<u>Example</u> $y = x^2$



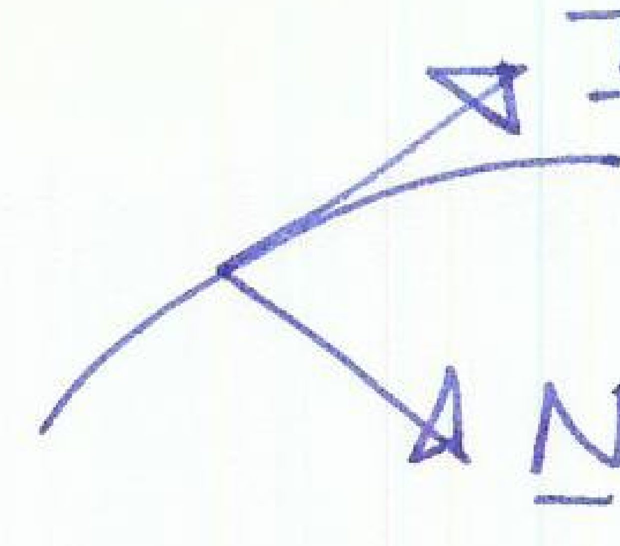
curvature is

$f(x) = x^2$

$f'(x) = 2x$ so $$\kappa(x) = \frac{2}{(1 + (2x)^2)^{3/2}}$$

$f(x) = 2$

Unit normal vector ([illegible])

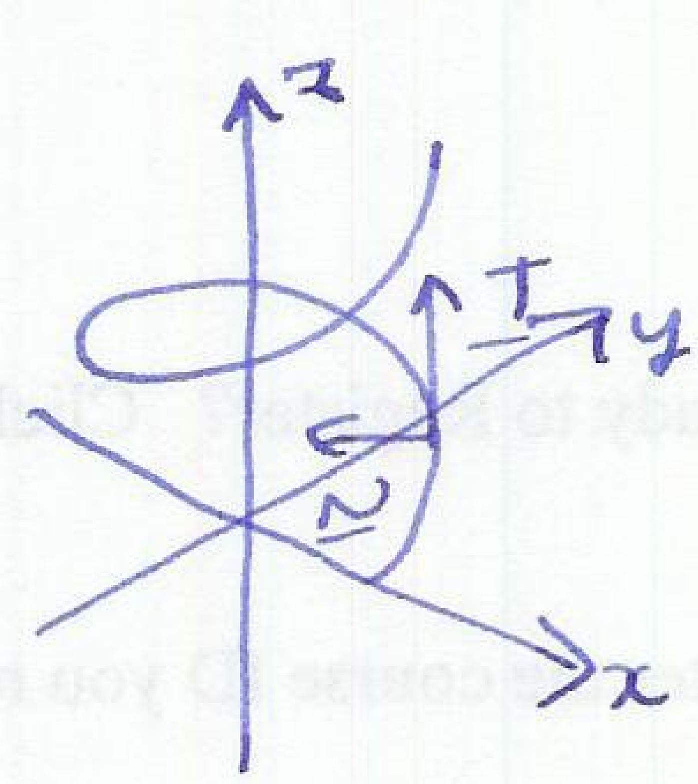


Def^n unit normal vector $\underline{N}(t) = \dfrac{\underline{T}'(t)}{\|\underline{T}'(t)\|}$

intuition: points in direction the curve is turning.

Example unit normal to a helix. $\underline{r}(t) = \langle \cos t, \sin t, t \rangle$

$\underline{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \qquad \|\underline{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$



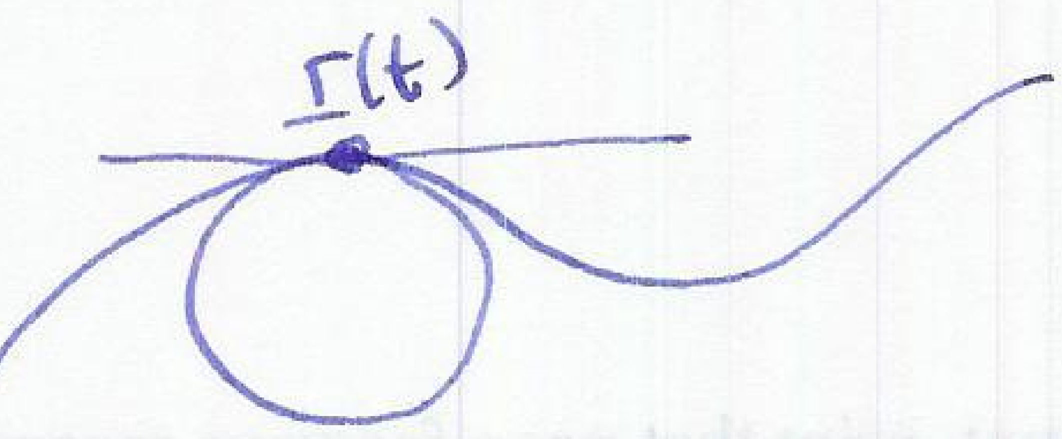
so $\underline{T}(t) = \dfrac{\underline{r}'(t)}{\|\underline{r}'(t)\|} = \dfrac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$

so $\underline{T}'(t) = \dfrac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$

so $\underline{N}(t) = \dfrac{\underline{T}'(t)}{\|\underline{T}'(t)\|} = \dfrac{\frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0 \rangle}{1} = -\dfrac{1}{\sqrt{2}} \langle \cos t, \sin t, 0 \rangle.$

Geometric interpretation of curvature

best fit circle:



circle which passes through $\underline{r}(t)$, and is tangent to $\underline{r}(t)$.

Fact radius of best fit circle is $\dfrac{1}{\kappa(t)}$.