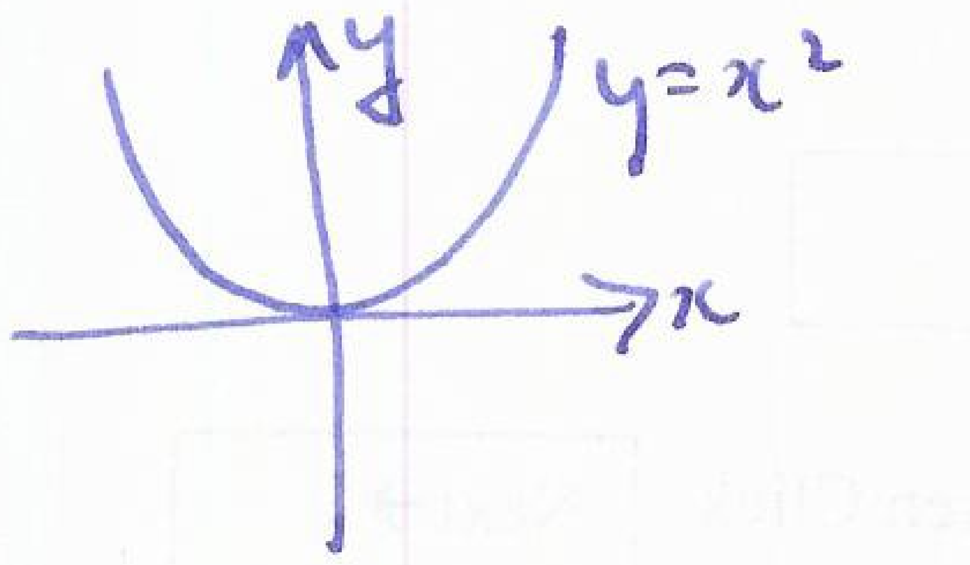
§13.1 Vector valued functions

real valued.
functions of 1-variable $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto f(x)$

parameterized curves / vector valued functions
of 1-variable $f: \mathbb{R} \to \mathbb{R}^2$ $t \mapsto \langle f(t), g(t) \rangle$
or $f: \mathbb{R} \to \mathbb{R}^3$ $t \mapsto \langle f(t), g(t), h(t) \rangle$

alternate notation: $t \mapsto \langle f(t), g(t) \rangle$
$\underline{r}(t) = f(t)\,\underline{i} + g(t)\,\underline{j}$

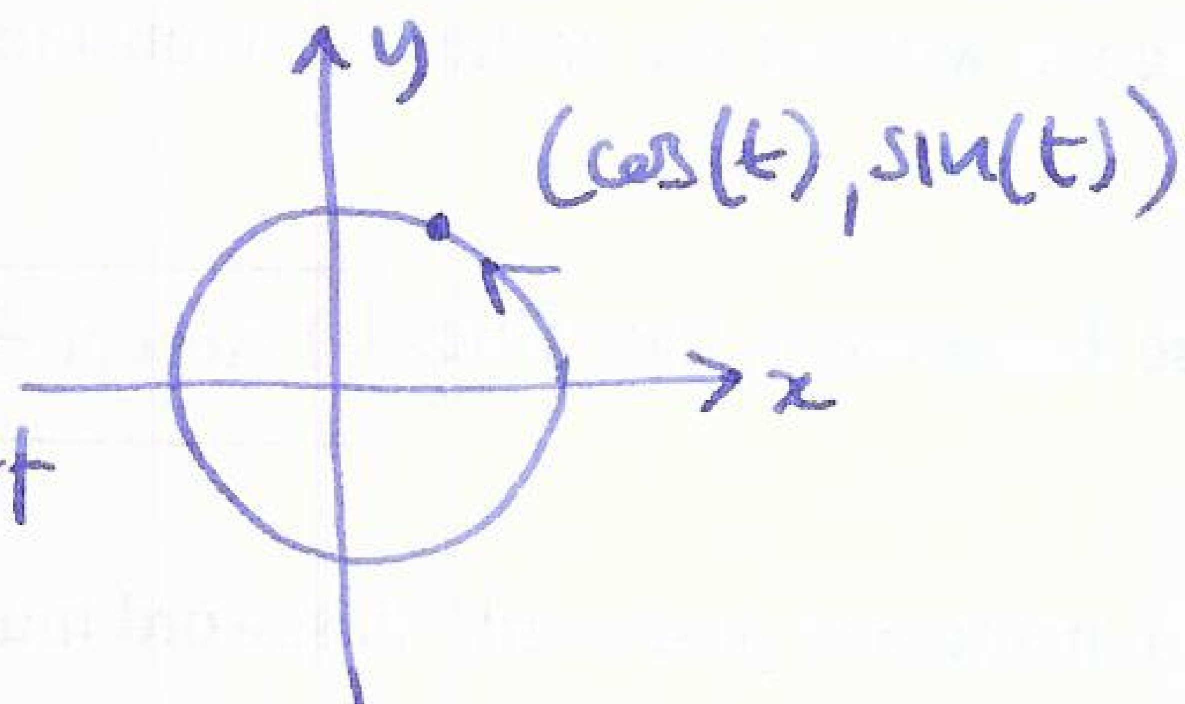
Example ① graph of a function of 1-variable.



$x \mapsto \langle x, x^2 \rangle$.

in general $x \mapsto \langle x, f(x) \rangle$.

② $\underline{r}(t) = \langle \cos(t), \sin(t) \rangle$



observation: $\left.\begin{matrix} x = \cos t \\ y = \sin t \end{matrix}\right\}$ $x^2+y^2 = \cos^2 t + \sin^2 t$
$= 1$.

warning: the same curve has many different parameterizations.

$\langle \cos t, \sin t \rangle$ $0 \le t \le 2\pi$

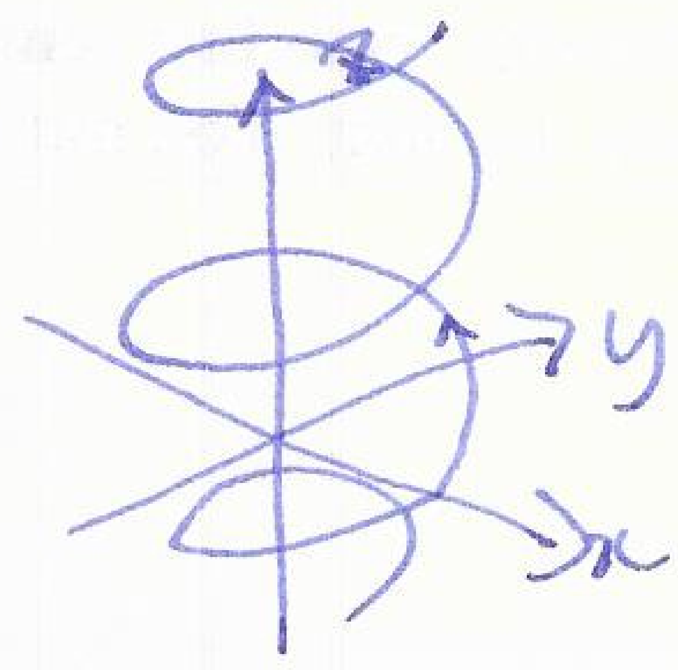
$\langle \cos 2t, \sin 2t \rangle$ $0 \le t \le \pi$

$\langle \cos t, \sin t \rangle$ $0 \le t \le \pi$
(goes halfway).

$\langle \cos 2t, \sin t \rangle$ $0 \le t \le 2\pi$
(goes around twice)

$\langle \cos t, \sin t \rangle$ $t \in \mathbb{R}$
goes round infinitely often.

$\underline{r}(t) = \langle \cos t, \sin t, t \rangle$ helix:

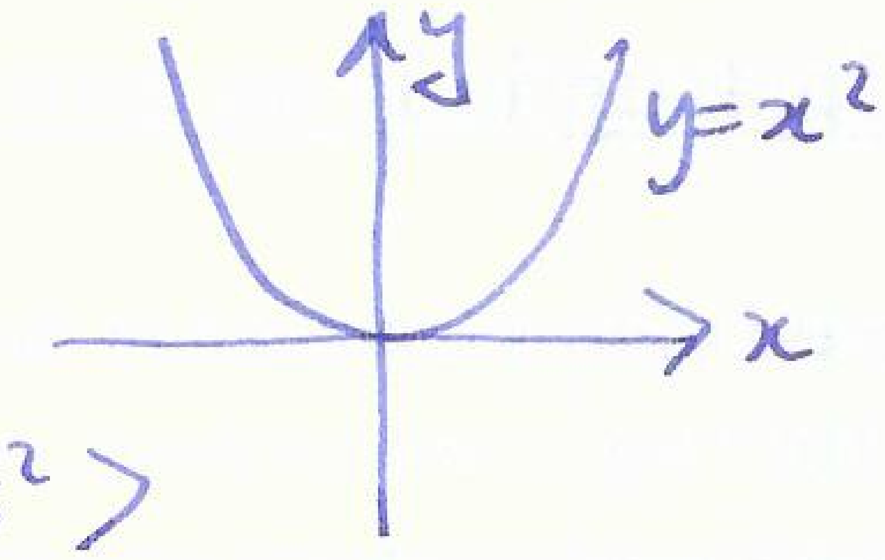


projections : $\langle \cos t, \sin t, 0 \rangle$ projection into xy-plane

$\langle 0, \sin t, 0 \rangle$ yz-plane

$\langle \cos t, 0, t \rangle$ xz-plane.

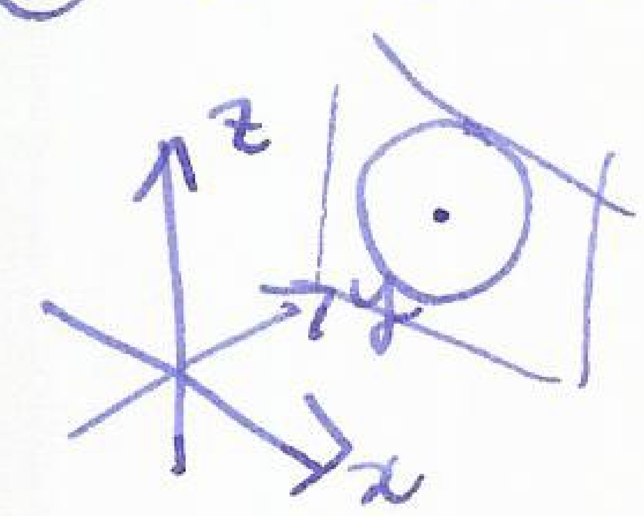
Finding a parameterization :



Example ① $y=x^2$ describes a plane curve :

set $x=t$, then $y=t^2$ so $r(t) = \langle t, t^2 \rangle$

② was circle of radius 4 in xz-plane with center $(1,2,3)$.



unit circle in xz-plane : $\underline{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$

radius 4 : $\underline{r}(t) = \langle 4\cos(t), 0, 4\sin(t) \rangle$

translate : $\underline{r}(t) = \langle 4\cos(t)+1, 2, 4\sin(t)+3 \rangle$.

Drawing parameterized curves

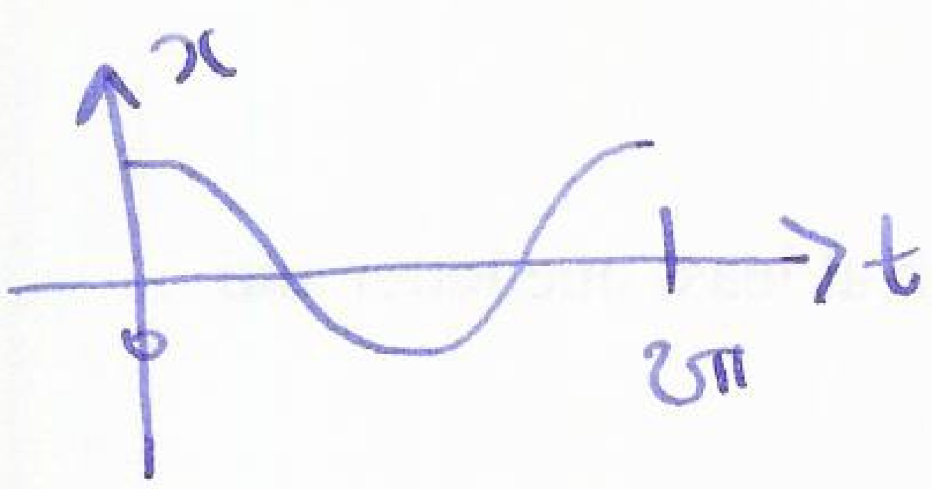
addition

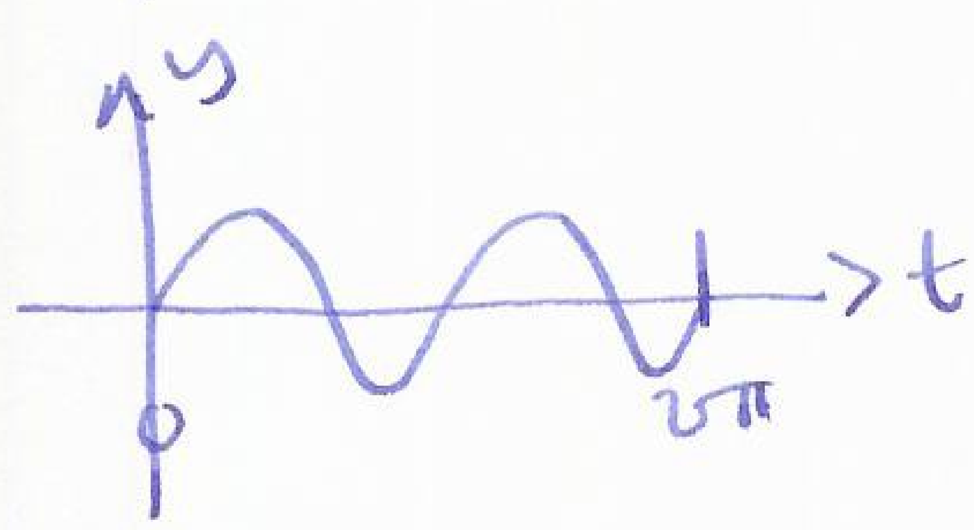
$\underline{r}(t) + \underline{s}(t)$

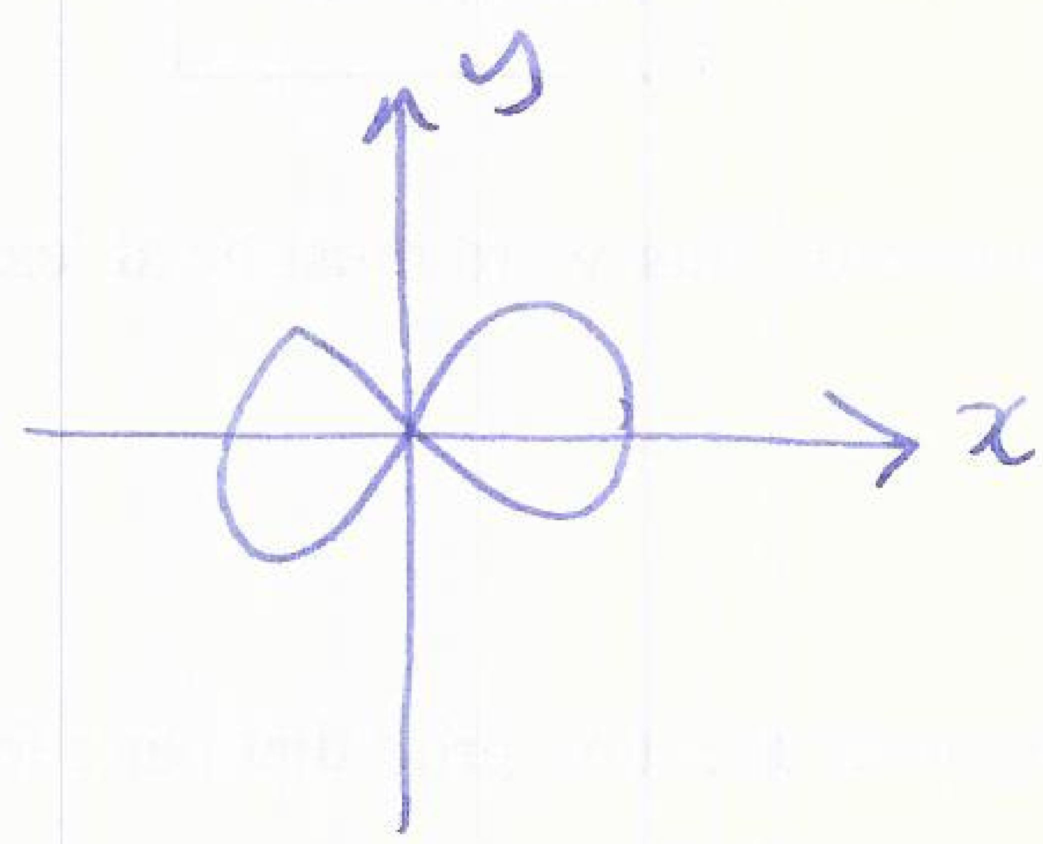
scalar multiplication

$\lambda \underline{r}(t)$.

Example $\underline{r}(t) = \langle \cos t, \sin 2t \rangle$







§13.2 Calculus of vector valued functions

limits : Def^{n} A vector valued function $\underline{r}(t)$ has a limit $\underline{v}$ at t_0 if $\lim_{t \to t_0} \| \underline{r}(t) - \underline{v} \| = 0$. If this happens, we write $\lim_{t \to t_0} \underline{r}(t) = \underline{v}$.

<u>Thm 1</u> Limits are computed componentwise.

$\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$ has a limit as $t \to t_0$ $\iff$ each component has a limit, and $\lim\limits_{t\to t_0} \underline{r}(t) = \langle \lim\limits_{t\to t_0} x(t), \lim\limits_{t\to t_0} y(t), \lim\limits_{t\to t_0} z(t) \rangle$.

<u>Proof</u> (sketch) $\|\underline{r}(t) - \underline{v}\|^2 = (x(t) - v_1)^2 + (y(t) - v_2)^2 + (z(t) - v_3)^2$.

LHS $\to 0 \iff$ each term in RHS $\to 0$. $\square$.

<u>Example</u> find $\lim\limits_{t\to 0} \underline{r}(t)$ where $\underline{r}(t) = \langle t^2, t, \frac{\sin t}{t} \rangle$.

$$\lim_{t\to 0} \underline{r}(t) = \langle \lim_{t\to 0} t^2, \lim_{t\to 0} t, \lim_{t\to 0} \frac{\sin t}{t} \rangle = \langle 0, 0, 1 \rangle.$$

<u>Continuity</u>:

<u>Defn</u> $\underline{r}(t)$ is continuous $\iff$ all of its components are continuous.

<u>Differentiation</u>:

<u>Defn</u> The derivative of $\underline{r}(t)$ is

$$\underline{r}'(t) = \frac{d\underline{r}}{dt} = \lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}$$

if this limit exists we say $\underline{r}(t)$ is <u>differentiable</u>.

<u>Thm 2</u> Derivatives are computed componentwise.

$\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$, then $\underline{r}'(t) = \frac{d\underline{r}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$.

<u>Proof</u> derivative defined as a limit, limits are computed componentwise $\square$

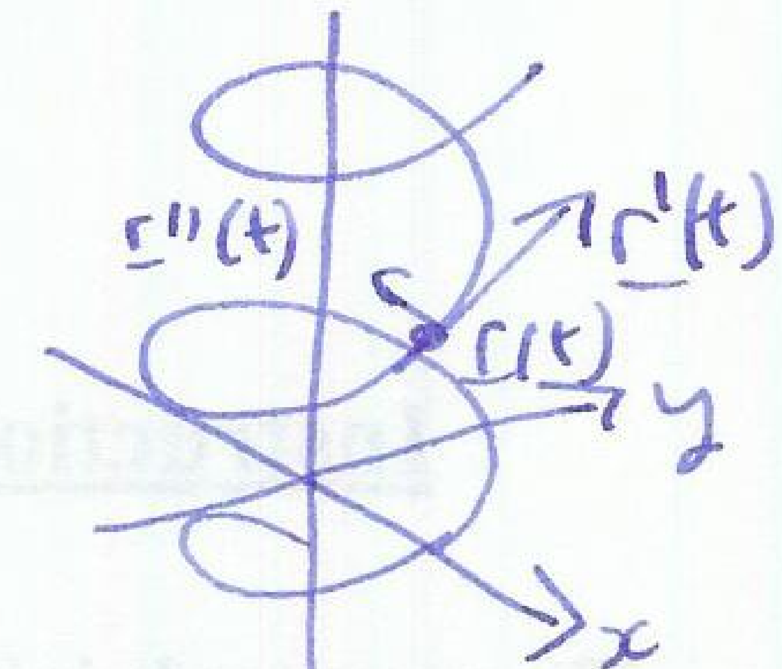
<u>Example</u> $\underline{r}(t) = \langle \cos t, \sin t, t \rangle$

$\underline{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

Higher derivatives: $\underline{r}''(t) = \frac{d^2\underline{r}}{dt^2} = \lim_{h\to 0} \frac{\underline{r}'(t+h) - \underline{r}'(t)}{h}$.

example:

so $\underline{r}''(t) = \langle -\cos t, -\sin t, 0 \rangle$.



if $\underline{r}(t)$ is position

$\underline{r}'(t)$ is velocity

$\underline{r}''(t)$ is acceleration

Differentiation rules (assume $\underline{r}(t)$, $\underline{r}_1(t)$, $\underline{r}_2(t)$ are differentiable)

- sum rule: $(\underline{r}_1(t) + \underline{r}_2(t))' = \underline{r}_1'(t) + \underline{r}_2'(t)$
- scalar mult: $(c\underline{r}(t))' = c\underline{r}'(t)$. c constant
- product rule: if $f(t)$, differentiable scalar function then

$$(f(t)\underline{r}(t))' = f(t)\underline{r}'(t) + f'(t)\underline{r}(t)$$

- chain rule: $g(t)$ differentiable scalar function

$$\frac{d}{dt}\left(\underline{r}(g(t))\right) = g'(t)\underline{r}'(g(t)).$$

Warning: $g(\underline{r}(t))$ doesn't make sense!

$\underline{r}_1(\underline{r}_2(t))$ "

Proof: (sketch) everything is defined componentwise.

product rule: $f(t)\underline{r}(t) = \langle f(t)x(t), f(t)y(t), f(t)z(t) \rangle$.

$$(f(t)\underline{r}(t))' = \langle (f(t)x(t))', (f(t)y(t))', (f(t)z(t))' \rangle.$$

$$= \langle f'x + fx', f'y + fy', f'z + fz' \rangle$$

$$= \langle f'x, f'y, f'z \rangle + \langle fx', fy', fz' \rangle$$

$$= f'\langle x, y, z \rangle + f\langle x', y', z' \rangle = f'(t)\underline{r}(t) + f(t)\underline{r}'(t).$$

others similarly $\square$.

<u>Example</u> $\underline{r}(t) = \langle t, t^2, t^3 \rangle \qquad f(t) = e^t$

$$\left(f(t)\underline{r}(t)\right)' = f'(t)\underline{r}(t) + f(t)\underline{r}'(t)$$

$$= e^t \langle t, t^2, t^3 \rangle + e^t \langle 1, 2t, 3t^2 \rangle$$

$$= e^t \langle 1+t, 2t+t^2, 3t^2+t^3 \rangle.$$

$$\left(\underline{r}(f(t))\right)' = \underline{r}'(f(t)) \cdot f'(t)$$

$$= \langle 1, 2e^t, 3e^{2t} \rangle \cdot e^t = \langle e^t, 2e^{2t}, 3e^{3t} \rangle.$$

<u>Thm 3</u> <u>Product rule for dot and cross products.</u>

$\underline{r}_1(t), \underline{r}_2(t)$ differentiable

<u>dot product</u>: $\frac{d}{dt}\left(\underline{r}_1(t) \cdot \underline{r}_2(t)\right) = \underline{r}_1(t) \cdot \underline{r}_2'(t) + \underline{r}_1'(t) \cdot \underline{r}_2(t)$

<u>cross product</u>: $\frac{d}{dt}\left(\underline{r}_1(t) \times \underline{r}_2(t)\right) = \underline{r}_1(t) \times \underline{r}_2'(t) + \underline{r}_1'(t) \times \underline{r}_2(t)$.

<u>Warning</u>: order matters in cross product!

<u>Proof</u> (sketch) dot, cross products defined componentwise

simple example in $\mathbb{R}^2$: $\frac{d}{dt}\left(\underline{r}_1(t) \cdot \underline{r}_2(t)\right) = \frac{d}{dt}\left(\langle x_1(t), y_1(t) \rangle \cdot \langle x_2(t), y_2(t) \rangle\right)$

$$= \frac{d}{dt}\left(x_1(t)x_2(t) + y_1(t)y_2(t)\right)$$

$$= x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2'$$

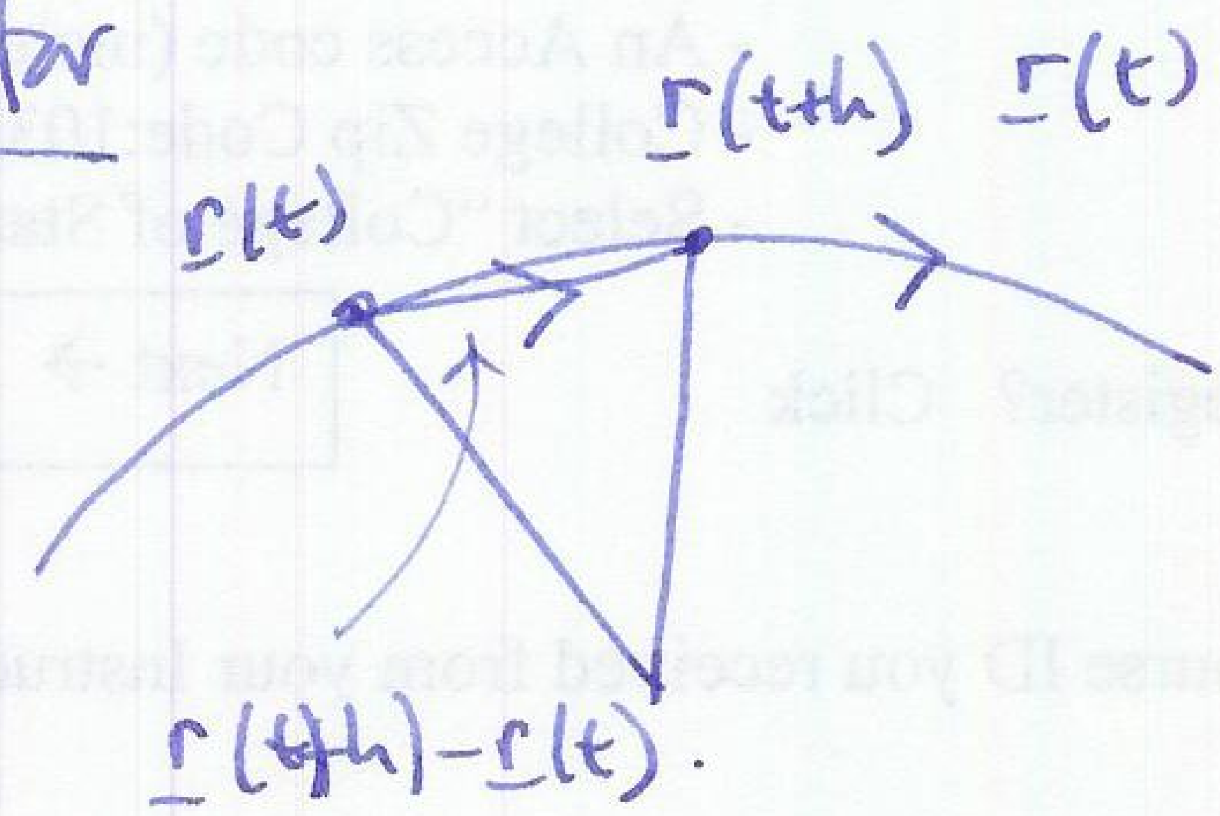
$$= x_1' x_2 + y_1' y_2 + x_1 x_2' + y_1 y_2'$$

$$= \underline{r}_1'(t)\,\underline{r}_2(t) + \underline{r}_1(t)\cdot \underline{r}_2'(t). \quad \square$$

Example show $\frac{d}{dt}\left(\underline{r}(t) \times \underline{r}'(t)\right) = \underline{r}(t) \times \underline{r}''(t)$.

$$\frac{d}{dt}\left(\underline{r}(t) \times \underline{r}'(t)\right) = \underbrace{\underline{r}'(t) \times \underline{r}'(t)}_{0} + \underline{r}(t) \times \underline{r}''(t)$$

The derivative is the tangent vector



$$\underline{r}'(t) = \lim_{h \to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}.$$

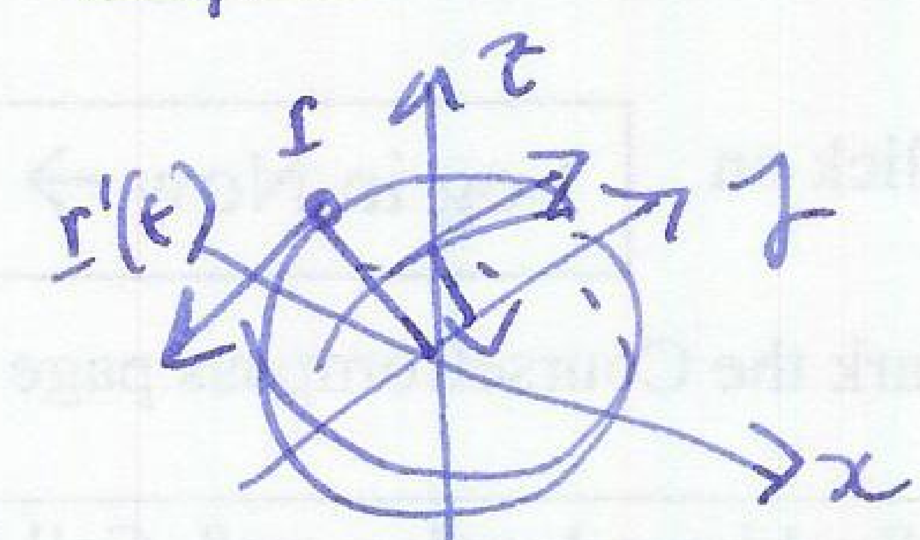
(or velocity vector)

Tangent line is given by: $\underline{L}(t) = \underline{r}(t_0) + t\,\underline{r}'(t_0)$

Example find tangent line to helix $\langle \cos t, \sin t, t \rangle$ at $t = 1$.

Example show $\underline{r}(t)$ and $\underline{r}'(t)$ are orthogonal if $\underline{r}(t)$ has unit length.

Explanation



unit length $\|\underline{r}(t)\|^2 = 1$

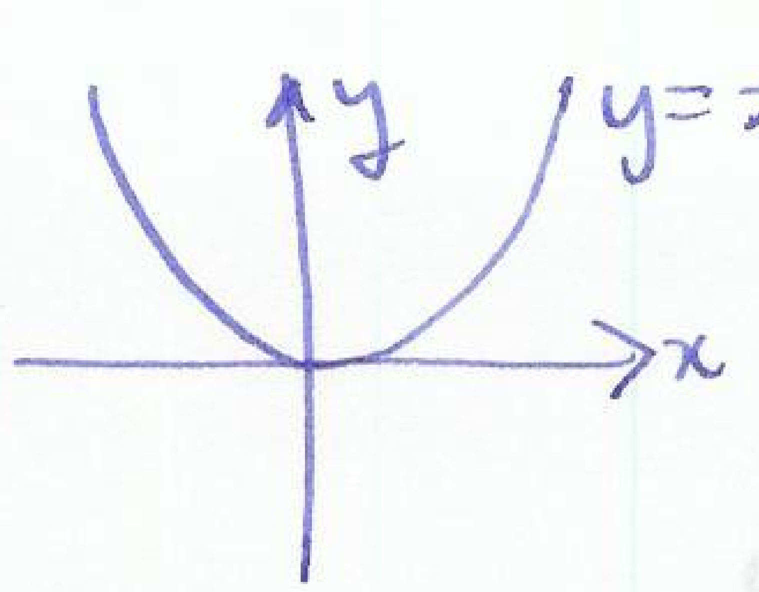
$$\| \quad \underline{r}(t)\cdot\underline{r}(t)$$

$$\frac{d}{dt}\left(\underline{r}(t)\cdot\underline{r}(t)\right) = \frac{d}{dt}(1) = 0$$

$$\| \quad \underline{r}'(t)\cdot\underline{r}(t) + \underline{r}(t)\cdot\underline{r}'(t) = 0$$

$$2\underline{r}(t)\cdot\underline{r}'(t) = 0 \Rightarrow \underline{r}, \underline{r}' \text{ orthogonal}.$$

<u>Example</u>



$\underline{r}(t) = \langle t, t^2 \rangle$.

$\underline{r}'(t) = \langle 1, 2t \rangle$

<u>note</u> $\|\underline{r}'(t)\| \neq 0$ for any t.

<u>Q</u>: what about $\underline{r}(t) = \langle t^3, t^6 \rangle$?

<u>Integration</u>

<u>Defn</u> We define integration to be componentwise, i.e.

$$\int_a^b \underline{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

the integral exists if each of the components is integrable

<u>Example</u>

$$\int_0^1 \langle t, t^2, \sin t \rangle\,dt = \left\langle \int_0^1 t\,dt, \int_0^1 t^2\,dt, \int_0^1 \sin t\,dt \right\rangle$$

$$= \left\langle \left[\tfrac{1}{2}t^2\right]_0^1, \left[\tfrac{1}{3}t^3\right]_0^1, \left[-\cos t\right]_0^1 \right\rangle$$

$$= \left\langle \tfrac{1}{2}, \tfrac{1}{3}, -\cos(1)+1 \right\rangle$$

<u>Antiderivatives</u>

<u>Defn</u> An antiderivative of $\underline{r}(t)$ is a function $\underline{R}(t)$ s.t. $\underline{R}'(t) = \underline{r}(t)$.

<u>Thm 4</u> Let $\underline{R}_1(t)$ and $\underline{R}_2(t)$ be antiderivatives of $\underline{r}(t)$ (ie $\underline{R}_1'(t) = \underline{R}_2'(t)$)

<u>Then</u> $\underline{R}_1(t) = \underline{R}_2(t) + \underline{c}$ $\quad \underline{c}$ constant vector.

<u>General antiderivative</u> $\quad \int \underline{r}(t)\,dt = \underline{R}(t) + \underline{c}$