note $\underline{v} \times \underline{v} = -\underline{v} \times \underline{v} = \underline{0}$

Thm Useful properties:

- $\underline{w} \times \underline{v} = -\underline{v} \times \underline{w}$
- $\underline{v} \times \underline{v} = \underline{0}$
- $\underline{v} \times \underline{w} = \underline{0} \iff \underline{w} = \lambda \underline{v}$ for some scalar $\lambda \in \mathbb{R}$ (or $\underline{v}$ or $\underline{w} = \underline{0}$)
- $(\lambda \underline{v}) \times \underline{w} = \underline{v} \times (\lambda \underline{w}) = \lambda (\underline{v} \times \underline{w})$
- $(\underline{u} + \underline{v}) \times \underline{w} = \underline{u} \times \underline{w} + \underline{v} \times \underline{w}$

  $\underline{u} \times (\underline{v} + \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w}$

Special case

$\underline{i} \times \underline{j} = \underline{k}$ $\quad$ $\underline{j} \times \underline{i} = -\underline{k}$ $\quad$ $\underline{i} \times \underline{i} = \underline{0}$ etc

$\underline{j} \times \underline{k} = \underline{i}$ $\quad$ $\underline{k} \times \underline{j} = -\underline{i}$

$\underline{k} \times \underline{i} = \underline{j}$ $\quad$ $\underline{i} \times \underline{k} = -\underline{j}$

Alternate way of computing $\underline{v} \times \underline{w}$

$\underline{v} = \langle 1, 0, 1 \rangle = \underline{i} + \underline{k}$

$\underline{w} = \langle 1, -1, 0 \rangle = \underline{i} - \underline{j}$

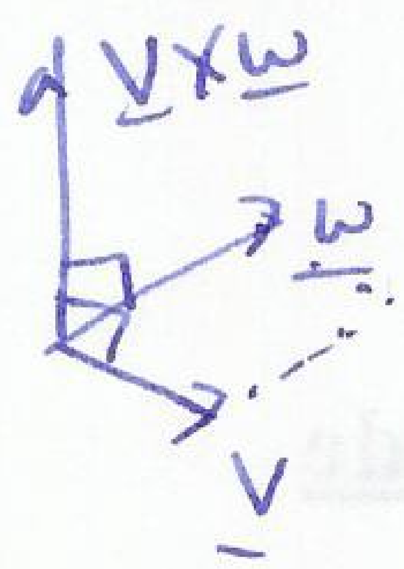
$\underline{v} \times \underline{w} = (\underline{i} + \underline{k}) \times (\underline{i} - \underline{j}) = \underline{i} \times (\underline{i} - \underline{j}) + \underline{k} \times (\underline{i} - \underline{j})$

$= \underline{i} \times \underline{i} - \underline{i} \times \underline{j} + \underline{k} \times \underline{i} - \underline{k} \times \underline{j}$

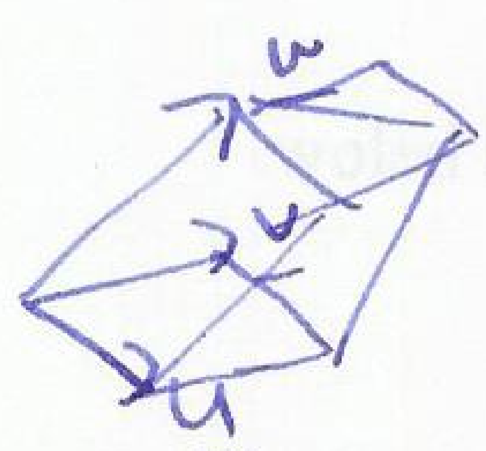
$= \underline{0} - \underline{k} + \underline{j} + \underline{i} = \underline{i} + \underline{j} - \underline{k}$

$= \langle 1, 1, -1 \rangle.$

Useful facts

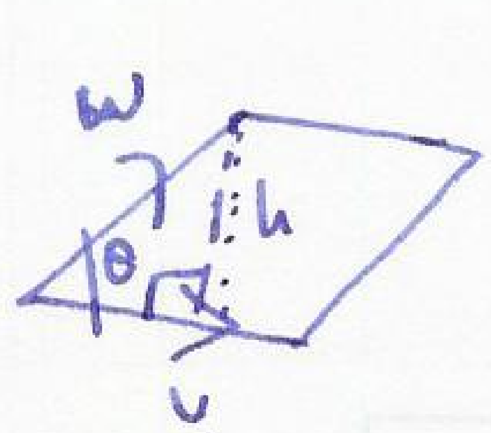


area of parallelogram defined by $\underline{v}, \underline{w}$ $\quad A = \| \underline{v} \times \underline{w} \|$.



volume of parallelopiped defined by $\underline{u}, \underline{v}, \underline{w}$ is $\quad V = | \underline{u} \cdot (\underline{v} \times \underline{w}) |$

check

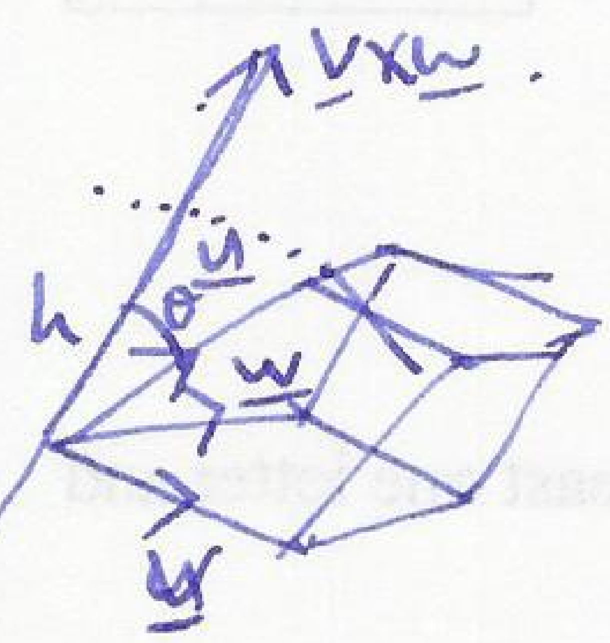


area of parallelogram is base $\times$ height

$$\|\underline{v}\| \times \|\underline{w}\| \sin\theta$$

$$= \| \underline{v} \times \underline{w} \|$$

volume of parallelopiped is



area of base $\times$ height

$$\| \underline{v} \times \underline{w} \| \quad \|\underline{u}\| \cos\theta$$

$$= | \underline{u} \cdot (\underline{v} \times \underline{w}) |$$

Notation $\quad \underline{u} \cdot (\underline{v} \times \underline{w})$ is called the vector triple product

Warning $\quad \underline{u} \cdot (\underline{v} \times \underline{w})$ makes sense

$(\underline{u} \cdot \underline{v}) \times \underline{w}$ does not!

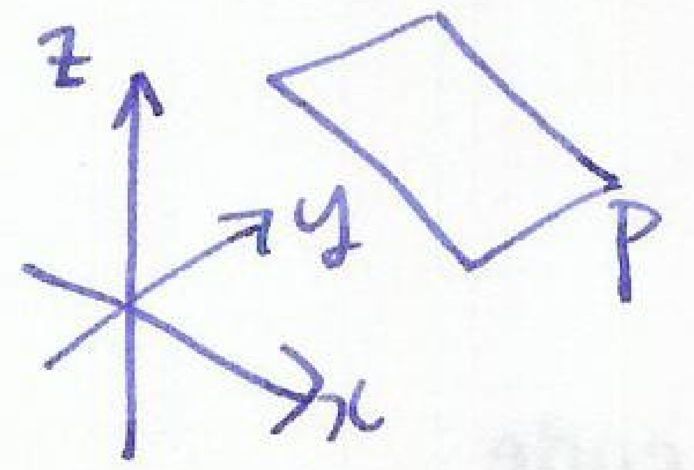
Note if $\underline{u} = \langle u_1, u_2, u_3 \rangle$ then $\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{u} \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$\underline{v} = \langle v_1, v_2, v_3 \rangle$

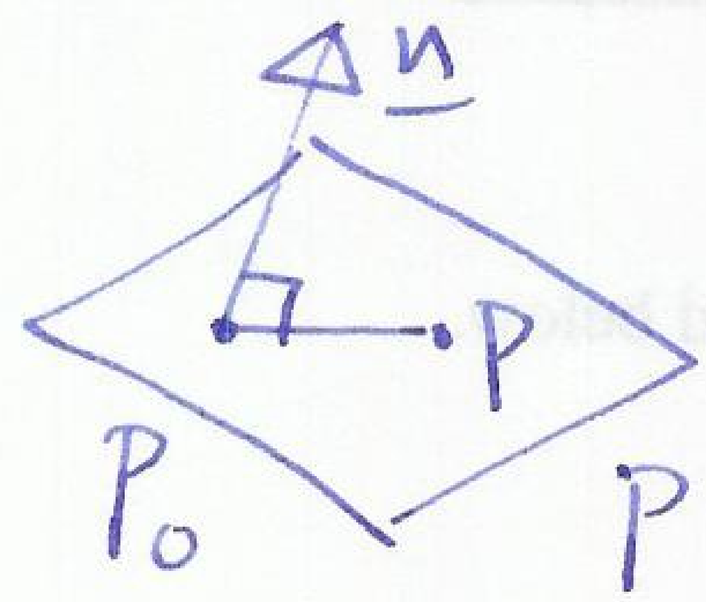
$\underline{w} = \langle w_1, w_2, w_3 \rangle$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \det \begin{pmatrix} \underline{u} \\ \underline{v} \\ \underline{w} \end{pmatrix}$$

§12.5 Planes in $\mathbb{R}^3$



claim: a plane in $\mathbb{R}^3$ is defined by a linear equation $ax+by+cz=d$



Proof let P_0 be a point on P $\quad P_0=(x_0,y_0,z_0)$

let $\underline{n}$ be the normal vector to P $\quad \underline{n}=\langle a,b,c\rangle$

let $P=(x,y,z)$ be some other point on the plane

then $\quad \underline{n}\cdot\overrightarrow{P_0P}=0$

$$\langle a,b,c\rangle\cdot\langle x-x_0,\ y-y_0,\ z-z_0\rangle=0$$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$ax+by+cz=\underbrace{ax_0+by_0+cz_0}_{=d}$$

Thm Equation of a plane in point-normal form

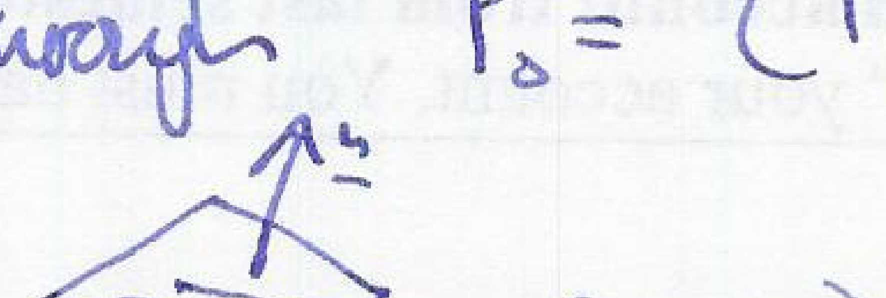
equation of plane through P_0 with normal vector $\underline{n}$ can be written as:

vector form: $\underline{n}\cdot\langle x,y,z\rangle=d \quad \left(d=\underline{n}\cdot\langle x_0,y_0,z_0\rangle = ax_0+by_0+cz_0\right)$

scalar form: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$ax+by+cz=d.$$

Example find equation of plane through $P_0=(\ ,2,3)$ with normal vector $\underline{n}=\langle 2,1,-1\rangle$

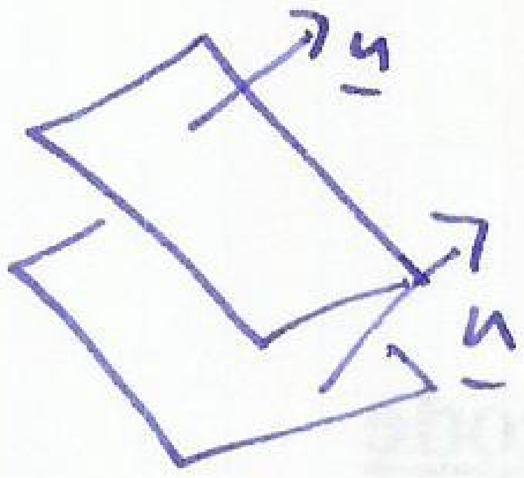


$\underline{n}\cdot\overrightarrow{P_0Q}=0$

$$\langle 2,1,-1\rangle\cdot\langle x-1,y-2,z-3\rangle = 2(x-1)+y-2-(z-3)=0$$

$$2x+y-z=7$$

• Parallel planes have the same normal vector



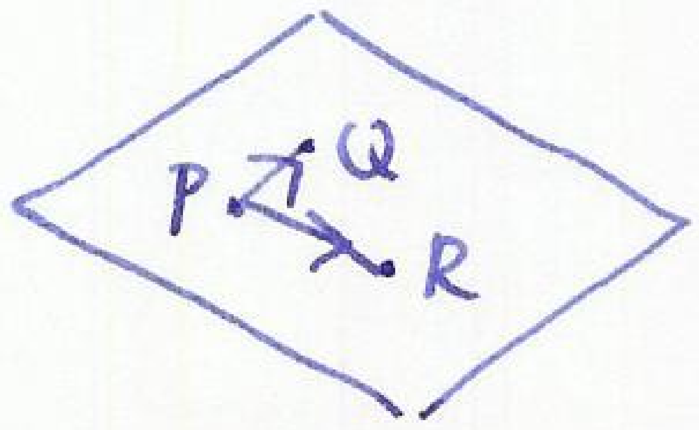
find the plane parallel to $x - 3y + 2z = 4$

through the point $P = (6, 4, 2)$

$\underline{n} = \langle 1, -3, 2 \rangle$

so equation of plane is $1(x-6) - 3(y-4) + 2(z-2) = 0$

• three points determine a plane.



find normal vector $\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

<u>Example</u> $P = (1, 0, 1)$ $Q = (2, 3, 1)$ $R = (-1, -1, 3)$

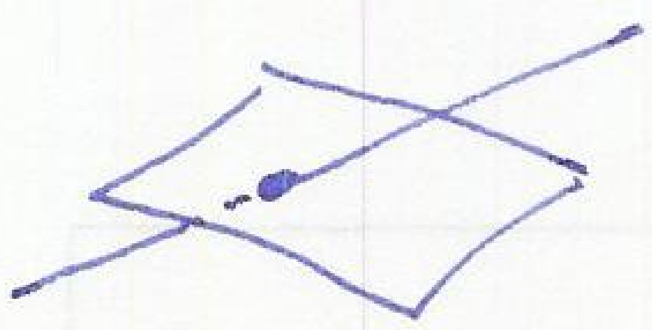
$\overrightarrow{PQ} = \langle 1, 3, 0 \rangle$ $\overrightarrow{PR} = \langle -2, -1, 2 \rangle$

$\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, 3, 0 \rangle \times \langle -2, -1, 2 \rangle = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 0 \\ -2 & -1 & 2 \end{vmatrix}.$

$= \langle 6, 2, -7 \rangle$

equation: $6(x-1) + 2(y-0) + -7(z-1) = 0.$

• intersection of a plane and a line



plane: $2x - 3y + 4z = 2$

line: $\langle 1, 2, 1 \rangle + t\langle -2, 1, 1 \rangle$

$x = 1 - 2t$

$y = 2 + t$

$z = 1 + t$

substitute in to equation of plane:

$$2(1-2t) - 3(2+t) + 4(1+t) = 2$$

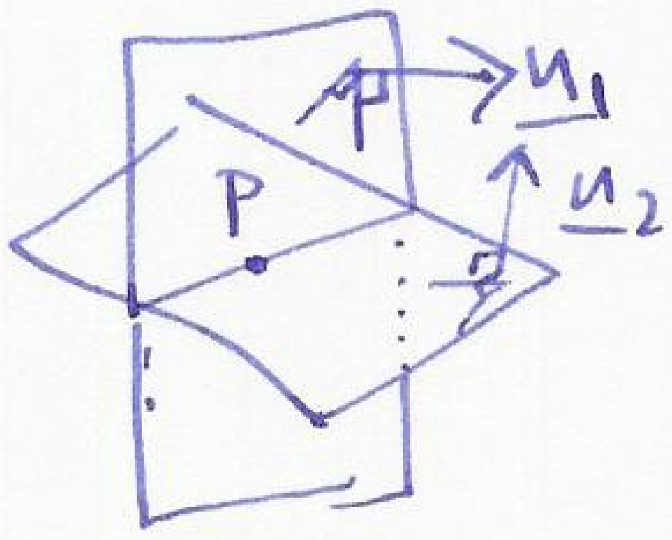
$$2 - 4t - 6 - 3t + 4 + 4t = 2$$

$$-3t = 2 \qquad t = 2/3.$$

so the point is: $\langle 1,2,1\rangle + \frac{2}{3}\langle -2,1,1\rangle = \langle -\frac{1}{3}, \frac{8}{3}, \frac{5}{3}\rangle$

check!

- intersection of two planes



direction vector for the line is perpendicular to both $\underline{n}_1, \underline{n}_2$ so can choose it to be

$$\underline{v} = \underline{n}_1 \times \underline{n}_2$$

then just need to find point on plane.

Example $\quad x+y-z=2 \qquad \underline{n}_1 = \langle 1,1,-1\rangle$

$\qquad x+2y+z=4 \qquad \underline{n}_2 = \langle 1,2,1\rangle$

$$\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \underline{i}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \langle -3, -2, 1\rangle.$$

find point on intersection

try: set $z=0$ $\quad \left.\begin{array}{l} ① \ x+y=2 \\ ② \ x+2y=4 \end{array}\right\}$ solve these

①−②: $-y=-2 \quad y=2 \implies x=0$ so $(0,2,0)$ works.