· unit vectors. $\underline{e}_v = \frac{1}{\|\underline{v}\|} \underline{v}$

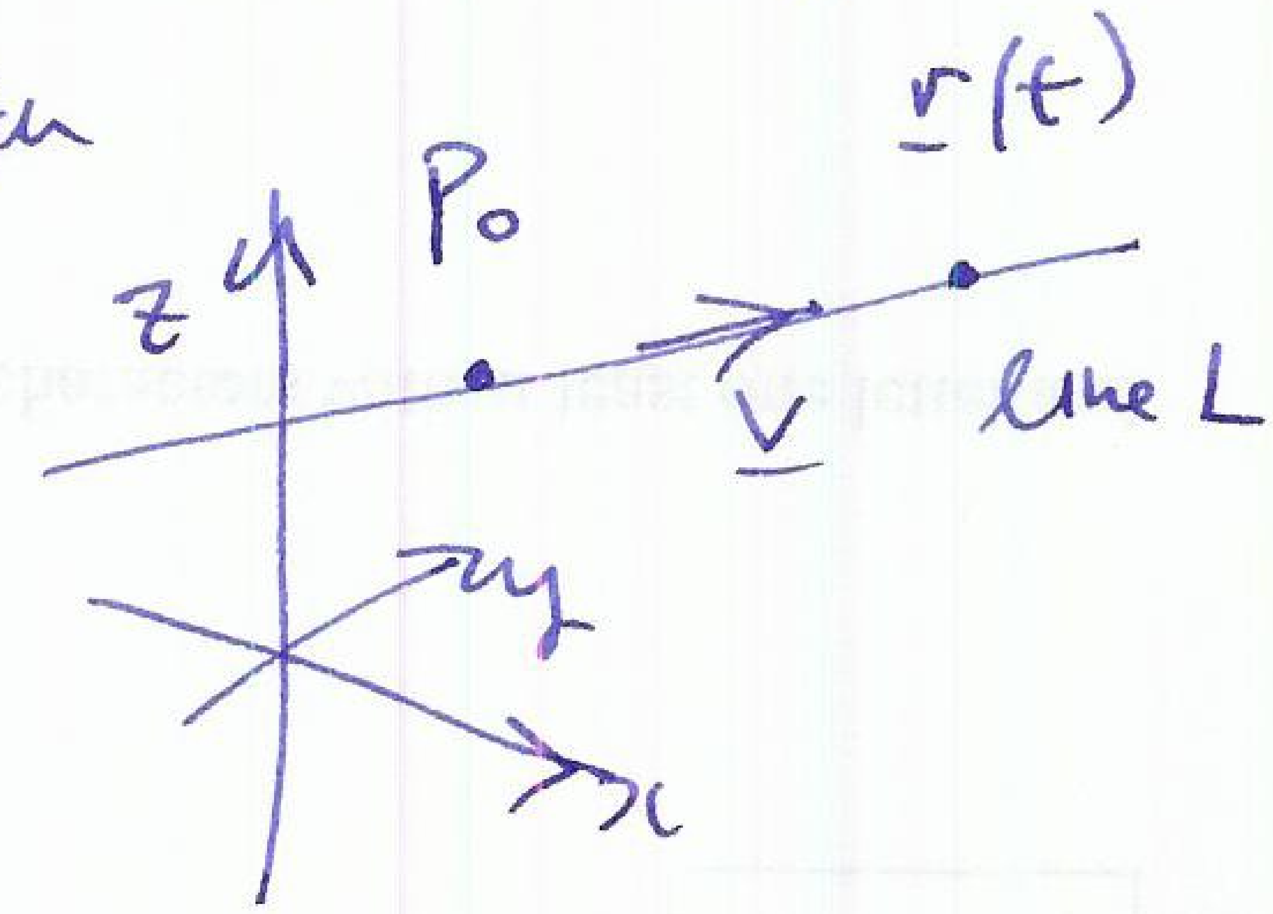
<u>example</u> $\underline{v} = \langle 1,2,3 \rangle \quad \|\underline{v}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$.

$\underline{e}_v = \frac{1}{\sqrt{14}} \langle 1,2,3 \rangle$.

<u>Equations of lines in $\mathbb{R}^3$</u>

$\mathbb{R}^2$: $y = mx+b$ $\quad$ $\mathbb{R}^3$: ~~$y = ax$~~ $ax+by+cz=d$ is a <u>plane</u>.

for lines in $\mathbb{R}^3$ use a parametric equation




a line $L$ is determined by

a point on $L$ : $P_0$

a direction : $\underline{v}$

any point on $L$ can then be written as $\underline{r}(t) = P_0 + t\underline{v}$.

($t$ is called the parameter). $\quad P_0 = \langle a,b,c \rangle$

$\underline{v} = \langle v_1, v_2, v_3 \rangle$

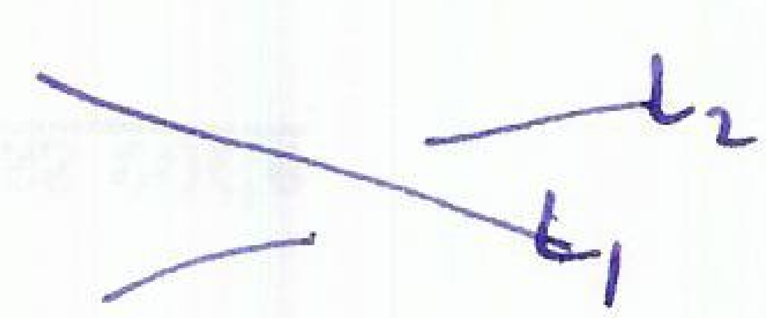
then $\underline{r} = \langle a+tv_1,\ b+tv_2,\ c+tv_3 \rangle$.

In $\mathbb{R}^2$ two lines $L_1, L_2$ are parallel or intersect.

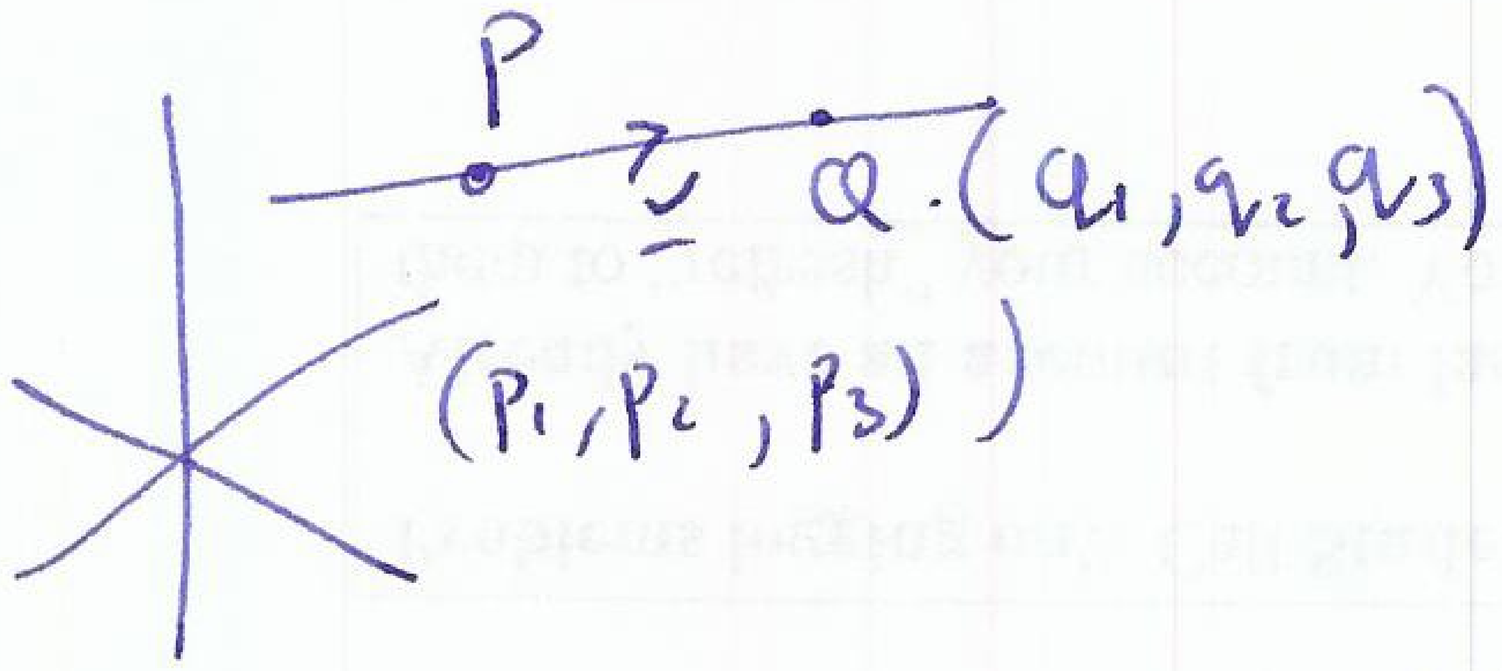
In $\mathbb{R}^3$ two lines $L_1, L_2$ may be parallel

intersect

neither (skew)



Parametric equation for a line through two points.




$\underline{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$

$\underline{r} = \underline{P} + t\underline{v}$.

observation : midpoint of $\overrightarrow{PQ}$ is $\underline{P} + \frac{1}{2}\cancel{(P-Q)}.\underline{v}$.

## §12.3 Dot product and angle

Def$^{\underline{n}}$ $\underline{v} = \langle v_1\ v_2\ v_3 \rangle$ $\quad \underline{w} = \langle w_1, w_2, w_3 \rangle$

then $\underline{v}.\underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.

Useful properties

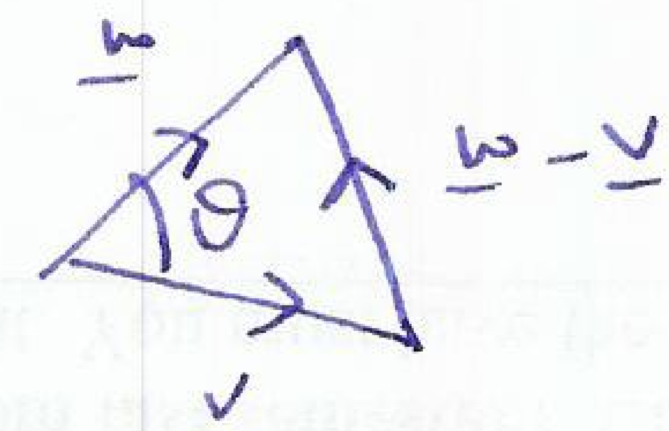
- $\underline{0}.\underline{v} = \underline{v}.\underline{0} = 0$
- commutativity $\underline{v}.\underline{w} = \underline{w}.\underline{v}$
- scalars : $(\lambda\underline{v}).\underline{w} = \underline{v}.(\lambda\underline{w}) = \lambda(\underline{v}.\underline{w})$
- distribution : $\underline{u}.(\underline{v}+\underline{w}) = \underline{u}.\underline{v} + \underline{u}.\underline{w}$
- length : $\underline{v}.\underline{v} = \|\underline{v}\|^2$

Angle between two vectors.


choose smallest angle
$0 \leq \theta \leq \pi$

Th$^{\underline{m}}$ $\underline{v}.\underline{w} = \|\underline{v}\|\,\|\underline{w}\| \cos\theta$ (or $\cos\theta = \dfrac{\underline{v}.\underline{w}}{\|\underline{v}\|\,\|\underline{w}\|}$

Proof (law of cosines)

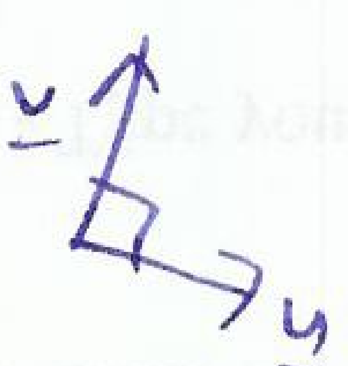



$$\|\underline{w}-\underline{v}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2 - 2\cos\theta\,\|\underline{v}\|\,\|\underline{w}\|$$

$$\|\underline{w}-\underline{v}\|^2 = (\underline{w}-\underline{v})\cdot(\underline{w}-\underline{v}) = \underline{w}\cdot\underline{w} - 2\underline{w}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$
$$= \|\underline{w}\|^2 - 2\underline{w}\cdot\underline{v} + \|\underline{v}\|^2.$$

So $-2\underline{w}\cdot\underline{v} = 2\cos\theta\,\|\underline{v}\|\,\|\underline{w}\| \Rightarrow \underline{w}\cdot\underline{v} = \|\underline{v}\|\|\underline{w}\|\cos\theta$. $\square$

Def$^{\underline{n}}$ $\underline{u},\underline{v}$ are perpendicular (or orthogonal) if $\underline{u}\cdot\underline{v}=0$

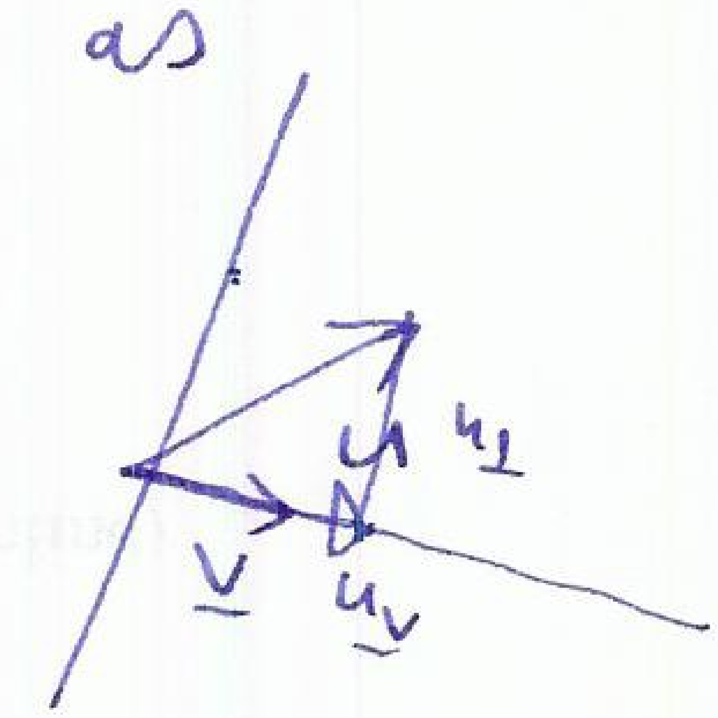



notation $\underline{u} \perp \underline{v}$

Projections given vectors $\underline{u},\underline{v}$ we can write $\underline{u}$ as

$\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$ where $\underline{u}_{\parallel}$ parallel to $\underline{v}$

$\underline{u}_{\perp}$ perpendicular to $\underline{v}$




$\underline{u}_{\parallel}$ is called the projection of $\underline{u}$ to $\underline{v}$ also written $\text{proj}_{\underline{v}}(\underline{u})$

Q: what is $\underline{u}_{\parallel}$?

A: $\underline{u}_{\parallel}$ is a multiple of $\underline{v}$. Let $e_{\underline{v}}$ be unit vector in direction of $\underline{v}$ then $\underline{u}_{\parallel} = c\underline{v}$ for some $c$

<u>note</u> $\underline{u}_\perp \cdot \underline{e}_v = 0$

$$\underline{u} \cdot \underline{e}_v = (\underline{u}_\parallel + \underline{u}_\perp) \cdot \underline{e}_v = \underline{u}_\parallel \cdot \underline{e}_v + \overbrace{\underline{u}_\perp \cdot \underline{e}_v}^{0} = c\underline{e}_v \cdot \underline{e}_v = c$$

so $$\underline{u}_\parallel = (\underline{u} \cdot \underline{e}_v) \cdot \underline{e}_v = \left(\underline{u} \cdot \frac{\underline{v}}{\|\underline{v}\|}\right) \frac{\underline{v}}{\|\underline{v}\|} = \frac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|^2} \underline{v} = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \underline{v}.$$

<u>Summary</u>

$$\boxed{\underline{u}_\parallel = \mathrm{proj}_{\underline{v}}(\underline{u}) = \underbrace{(\underline{u} \cdot \underline{e}_v)}_{} \underline{e}_v = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \underline{v}}$$

this is called <u>the component</u> of $\underline{u}$ along $\underline{v}$.

<u>Example</u>.

## <u>§12.4 Cross product</u>

**Example** $\underline{u} = \langle 1, 2, 3 \rangle$

$\underline{v} = \langle 1, -1, 1 \rangle$

write $\underline{u}$ as $\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$ where $\underline{u}_{\parallel}$ parallel to $\underline{v}$

$\underline{u}_{\perp}$ perpendicular to $\underline{v}$

$$\underline{u}_{\parallel} = \text{proj}_{\underline{v}}(\underline{u}) = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \underline{v} = \frac{1 \cdot 1 + 2 \cdot (-1) + 3 \cdot 1}{1 + 1 + 1} = \frac{2}{3} \underline{v}.$$

so $\underline{u}_{\parallel} = \langle \frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$

$$\underline{u}_{\perp} = \underline{u} - \underline{u}_{\parallel} = \langle 1, 2, 3 \rangle - \frac{2}{3} \langle 1, -1, 1 \rangle = \langle \tfrac{1}{3}, 2\tfrac{2}{3}, 2\tfrac{1}{3} \rangle.$$

**check** $\underline{v} \cdot \underline{u}_{\perp} = \langle 1, -1, 1 \rangle \cdot \langle \frac{1}{3}, \frac{8}{3}, \frac{7}{3} \rangle = \frac{1}{3} - \frac{8}{3} + \frac{7}{3} = 0.$

## §12.4 Cross products

dot product $\underline{v} \cdot \underline{w}$ ← scalar!

cross product $\underline{v} \times \underline{w}$ ← vector!

**Recall**: A $2 \times 2$ matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

**Example** $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1.$

A $3 \times 3$ matrix $\begin{bmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

<u>Example</u>

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 1(-1) - 2(-1) + 3(-1) = -1+2-3 = -2.$$

<u>Def$^{\underline{n}}$</u> if $\underline{v} = \langle v_1, v_2, v_3 \rangle$ $\quad \underline{w} = \langle w_1, w_2, w_3 \rangle$

then $$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \underline{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \underline{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \underline{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \langle v_2 w_3 - v_3 w_2, \; -v_1 w_3 + w_1 v_3, \; v_1 w_2 - w_1 v_2 \rangle$$

<u>Example</u> $\underline{v} = \langle 1, 2, 3 \rangle$ $\quad \underline{w} = \langle 1, -1, -1 \rangle$

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} = \underline{i}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \langle -1, 4, -3 \rangle.$$

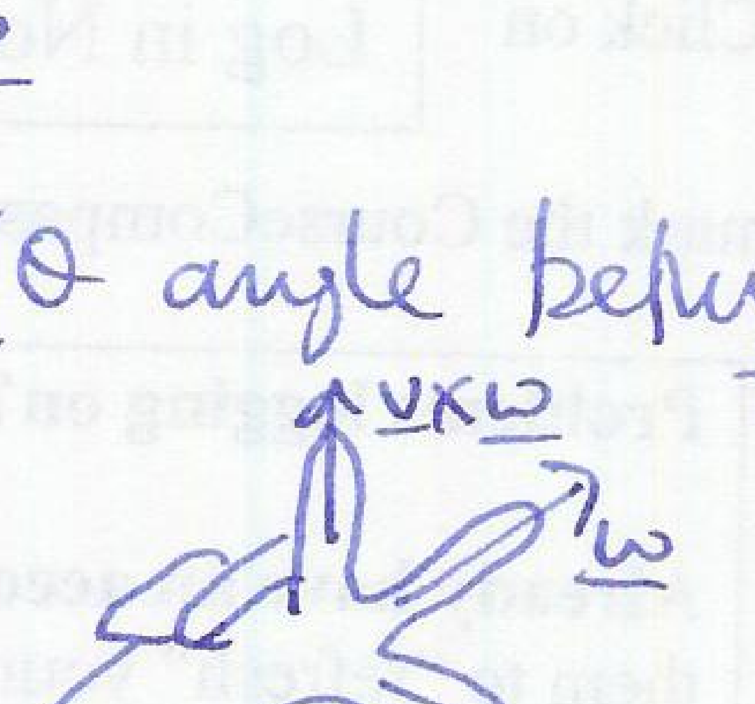
<u>Th$^{\underline{m}}$</u> Geometric description of cross product

$\underline{v} \times \underline{w}$ is the unique vector with the following properties

i) $\underline{v} \times \underline{w}$ is perpendicular to both $\underline{v}$ and $\underline{w}$

ii) length of $\underline{v} \times \underline{w}$ is $\|\underline{v}\|\|\underline{w}\| |\sin\theta|$ ($\theta$ angle between them)

iii) $\{\underline{v}, \underline{w}, \underline{v} \times \underline{w}\}$ is right-handed




<u>Warning</u> $\underline{v} \times \underline{w} = -\underline{w} \times \underline{v}$ (anti-commutative!)