Math 233 Calculus III

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prerequisites: Calc I, II: derivatives, integrals etc.

- Math tutoring: 1S-214
- students with disabilities

Text: Early Transcendentals, by Rogawski

§12.1 2d vectors

scalar/number

size/magnitude only

examples: 7
-4.3
etc.

examples: length
temperature
pressure
time

speed = length of velocity vector

notation: $7, \pi \in \mathbb{R}$
$s, t \in \mathbb{R}$

vector

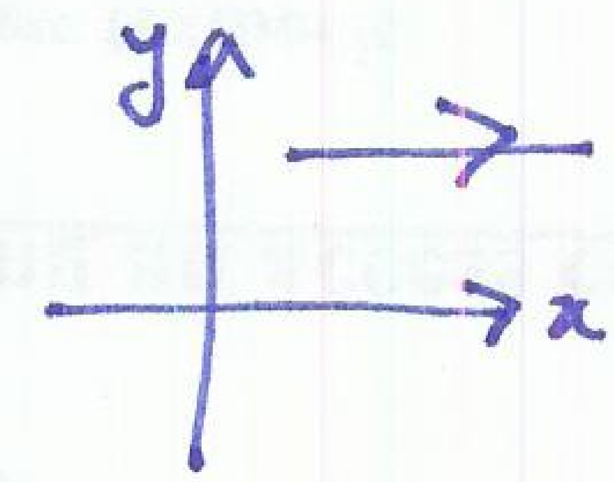
size and direction

examples: $\downarrow$, $\nearrow$ ← length and direction

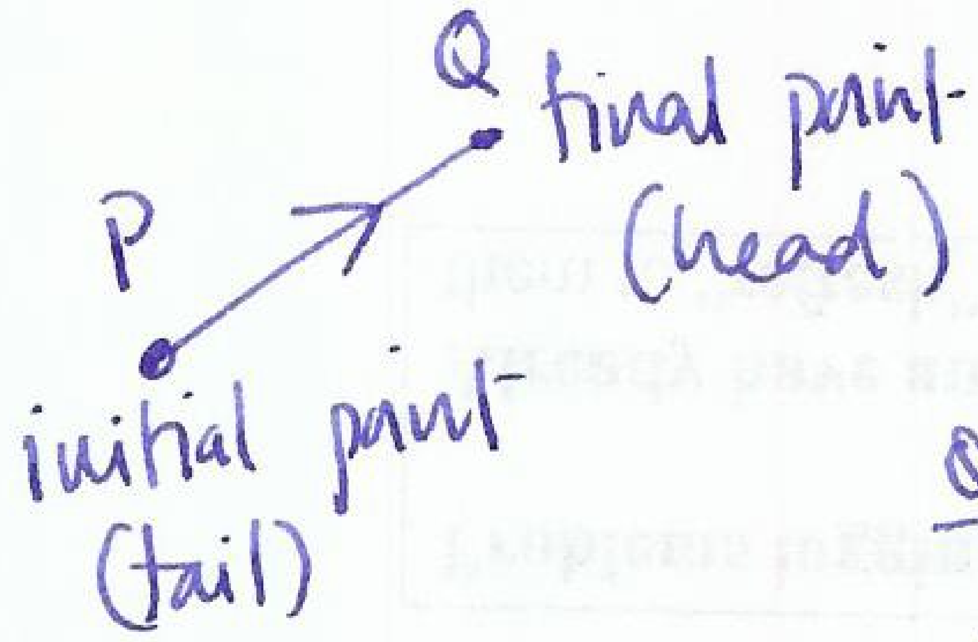
examples: force
velocity.

notation: $\underline{v}$ $\vec{v}$

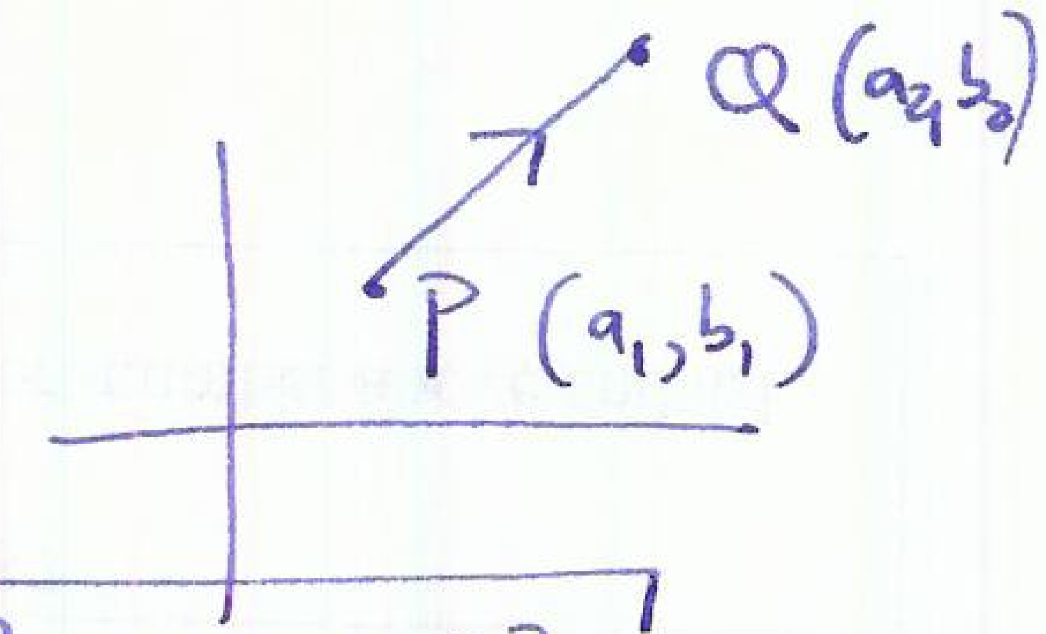
"length 4 in direction of x-axis"



a vector $\underline{v}$ is determined by its initial and final points.



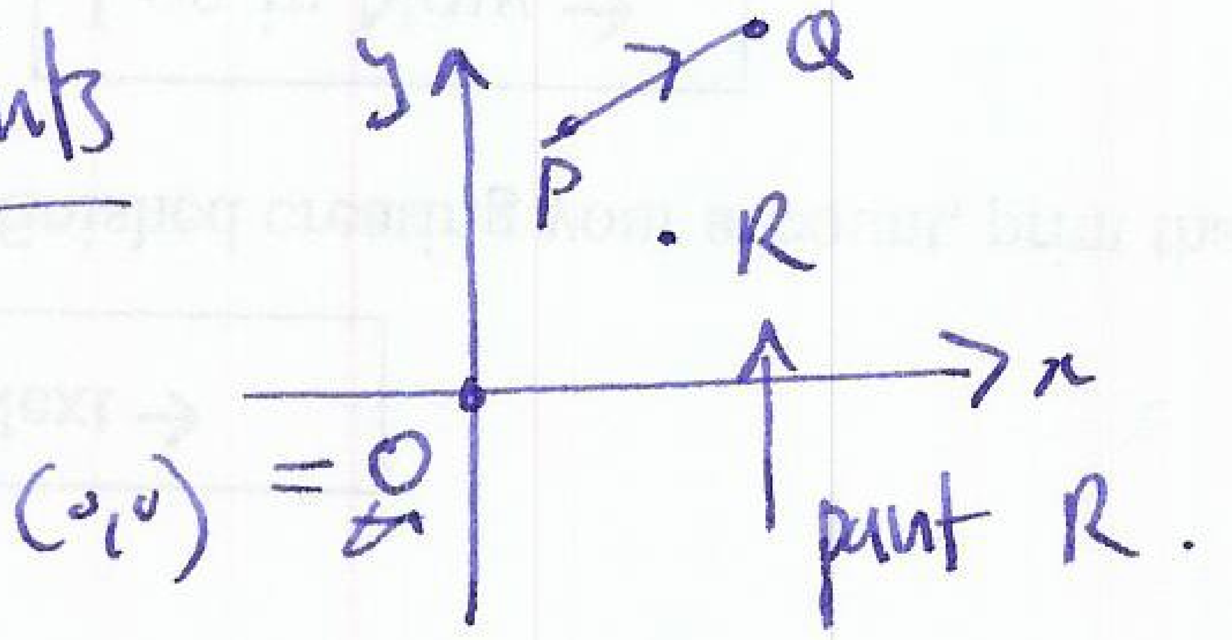
notation $\underline{v} = \overrightarrow{PQ}$



Q: how long is the vector?

A: $\|\underline{v}\| = \|\overrightarrow{PQ}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$.

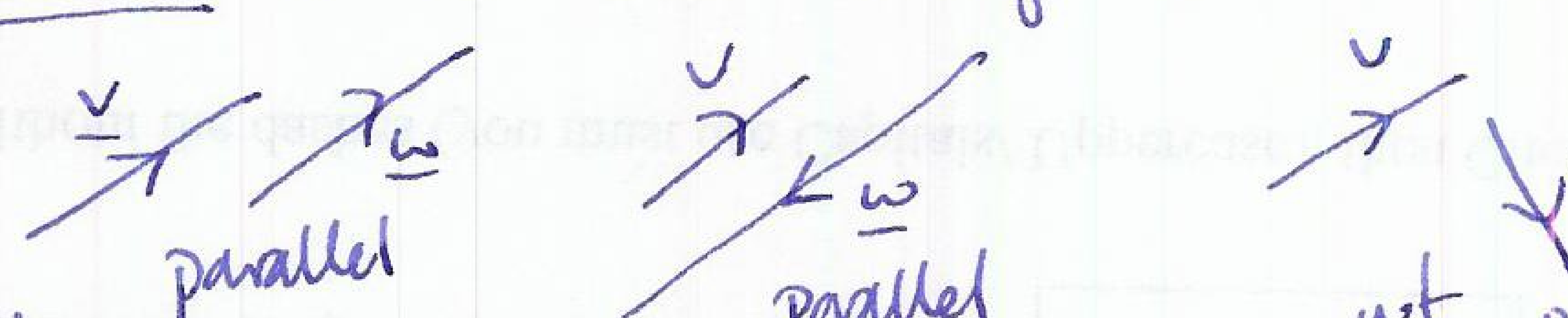
vectors vs points



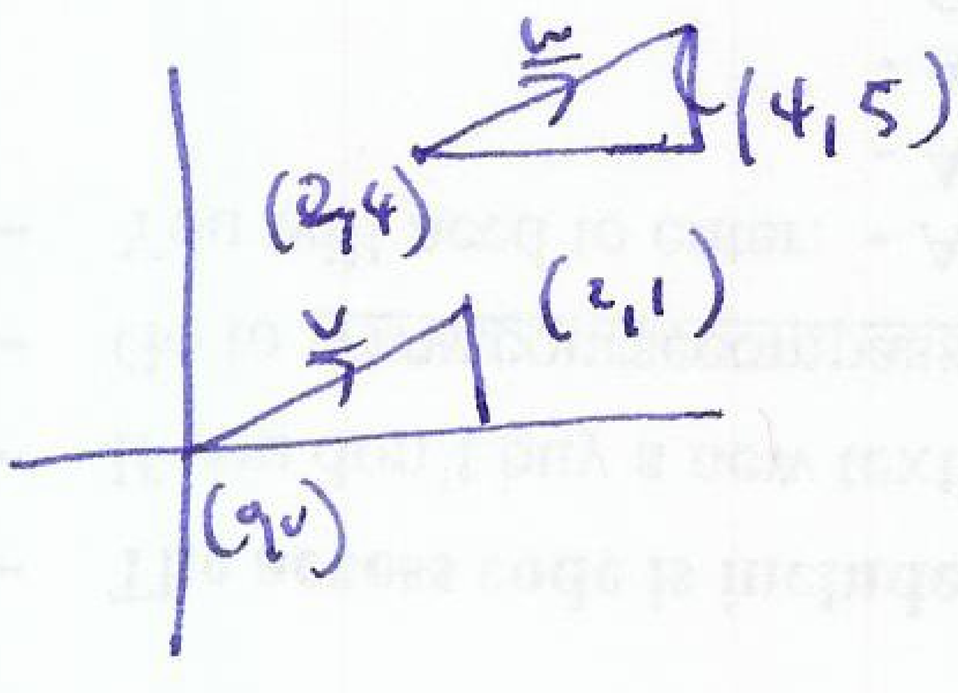
every point R corresponds to a special position vector $\overrightarrow{OR}$

- two vectors, $\underline{v}$ and $\underline{w}$, are parallel if the lines through $\underline{v}$ and $\underline{w}$ have the same direction

i.e. $\underline{v}, \underline{w}$ have same or opposite direction

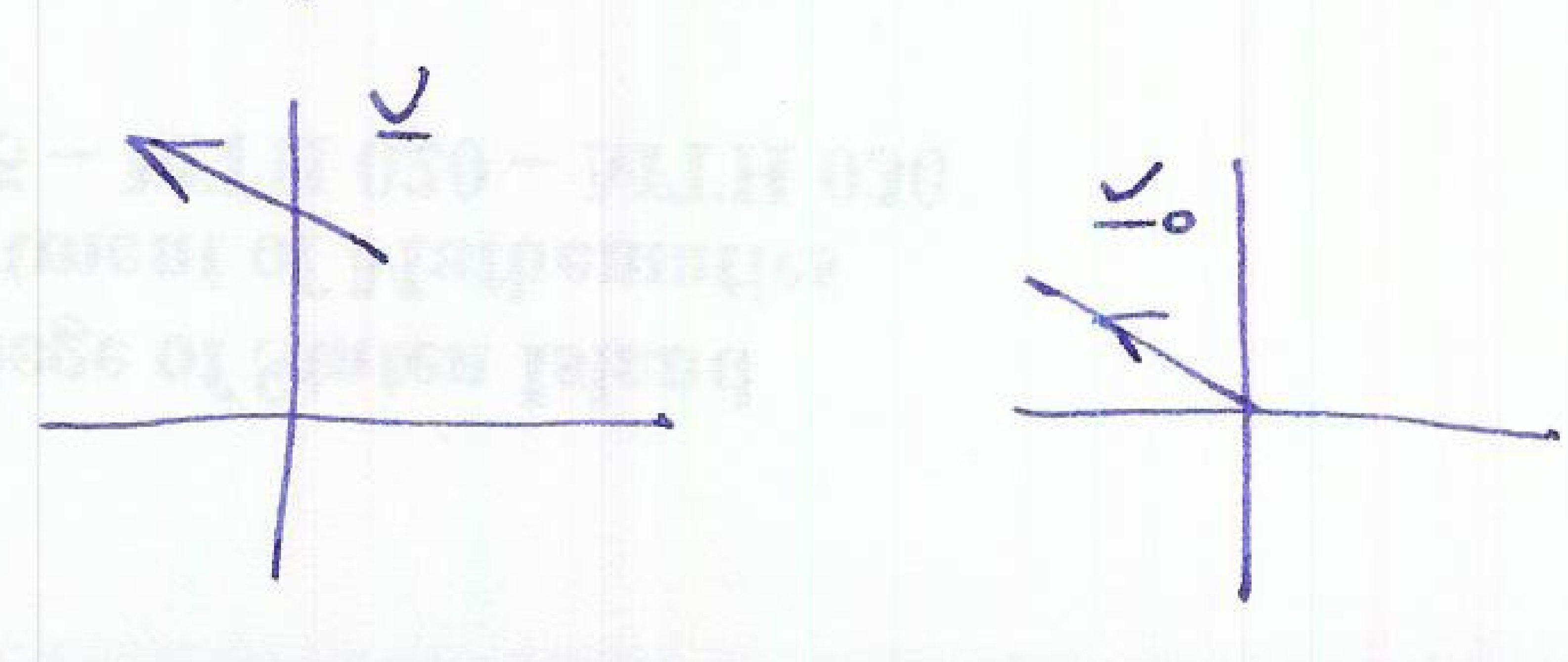


- two vectors $\underline{v}, \underline{w}$ are translates if they have same (not opposite) direction and length, but different base points.

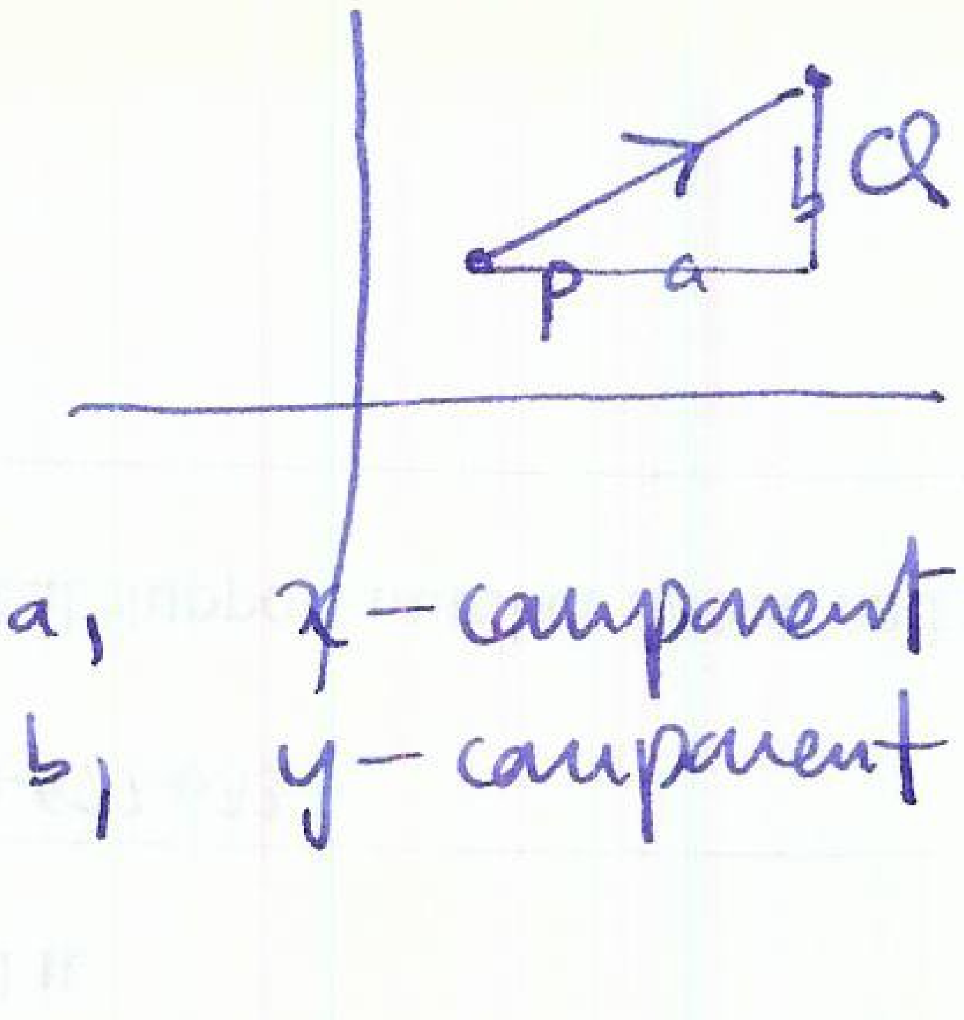


we say $\underline{v}$ and $\underline{w}$ are equivalent (or equal) if $\underline{v}$ and $\underline{w}$ are translates of each other.

Observation: every vector $\underline{v}$ is equivalent to a unique vector $\underline{v}_0$ based at the origin $\underline{0}$



Def$^{\underline{n}}$ components of a vector. $\underline{v} = \overrightarrow{PQ}$



where $P = (a_1, b_1)$ $\quad Q = (a_2, b_2)$

then the components of $\underline{v}$ are $a = a_2 - a_1$ x-component
$b = b_2 - b_1$ y-component

notation: $v = \langle a, b \rangle$

Observation: • $\|\underline{v}\| = \sqrt{a^2 + b^2}$

• components $\langle a, b \rangle$ determine length and direction

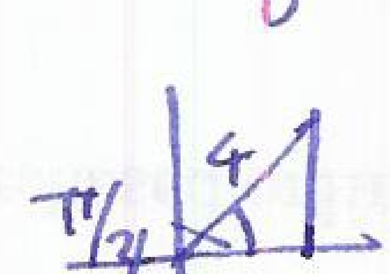
so two vectors are equivalent $\Leftrightarrow$ they have the same components.

• $\underline{v} = \langle a, b \rangle$ does not determine basepoint.

Convention: all vectors are based at the origin O unless otherwise stated.

Special vector: $\underline{0} = \langle 0, 0 \rangle$. zero vector.

Example A vector $\underline{v}$ has length 4, lies in 1st quadrant and makes an angle of 45° with the x-axis. Find the components of $\underline{v}$



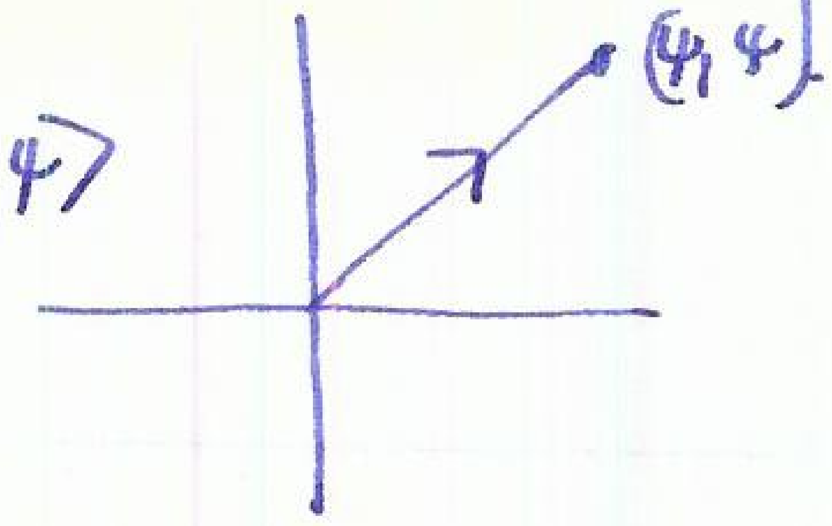
Vector addition and scalar multiplication

scalar multiplication: λ scalar / number
$\underline{v}$ vector

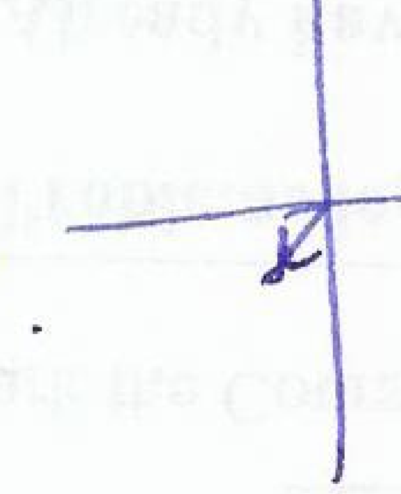
$\lambda \underline{v}$ is vector with same direction as $\underline{v}$, if $\lambda > 0$ but length $\lambda \|\underline{v}\|$.
opposite if $\lambda < 0$ $\quad -\lambda \|\underline{v}\|$.

Example $\underline{v} = \langle 1,1 \rangle$

$4\underline{v} = \langle 4,4 \rangle$

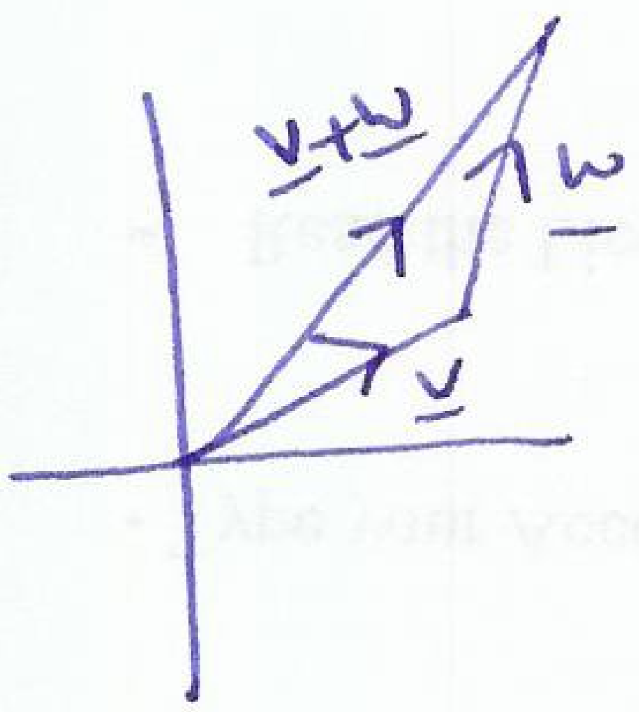


$-\frac{1}{2}\underline{v} = \langle -\frac{1}{2}, -\frac{1}{2} \rangle$.

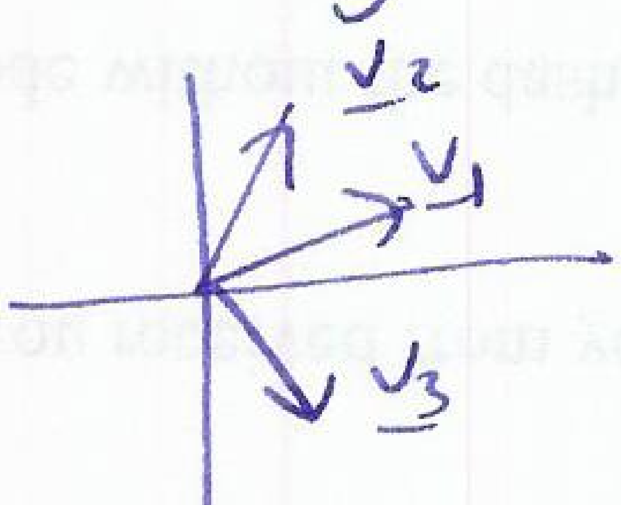


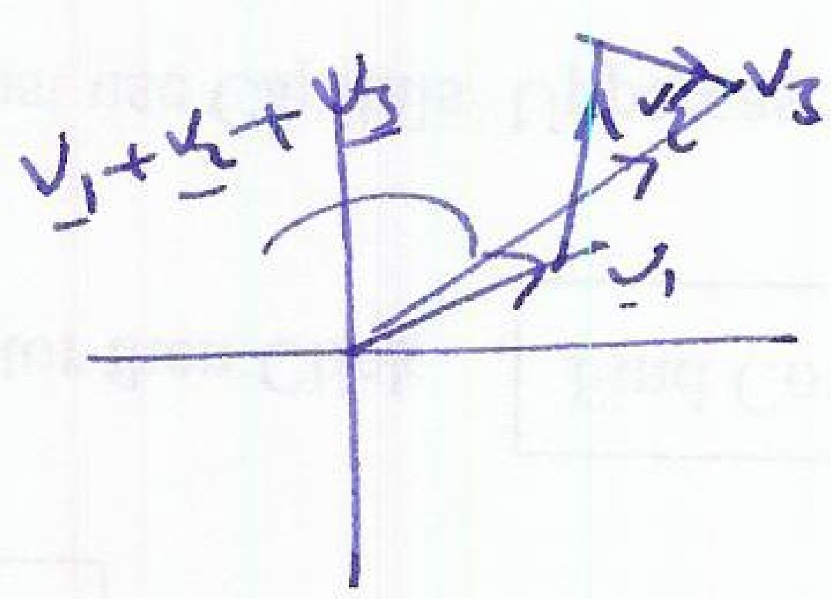
observation: $\underline{v}$ is parallel to $\underline{w}$ iff $\underline{v} = \lambda \underline{w}$ for some λ.

vector addition: $\underline{v}, \underline{w}$ vectors. $\underline{v}+\underline{w}$: translate $\underline{w}$ so initial point of $\underline{w}$ is terminal point of $\underline{v}$. Then $\underline{v}+\underline{w}$ is vector from initial point of $\underline{v}$ to terminal point of $\underline{w}$

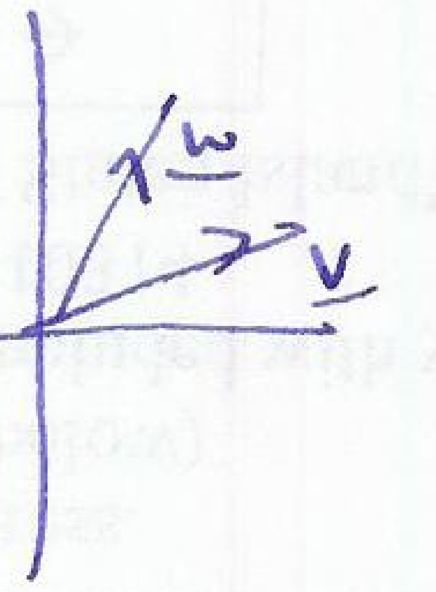


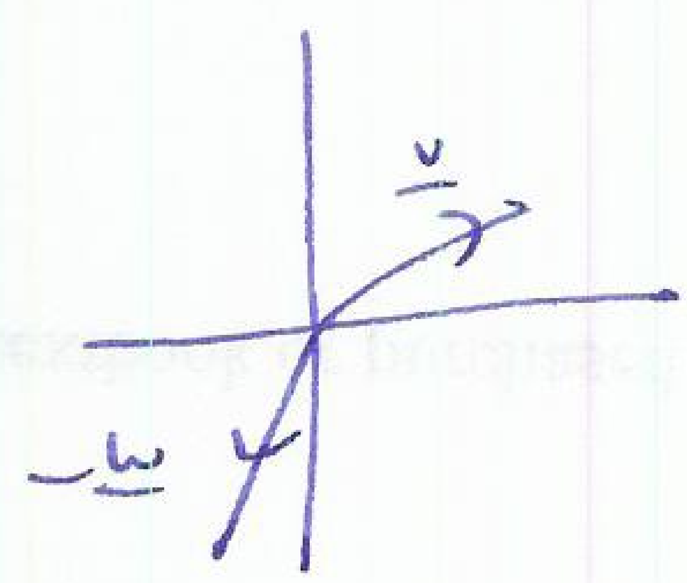
- adding multiple vectors $\underline{v}_1 + \underline{v}_2 + \underline{v}_3$

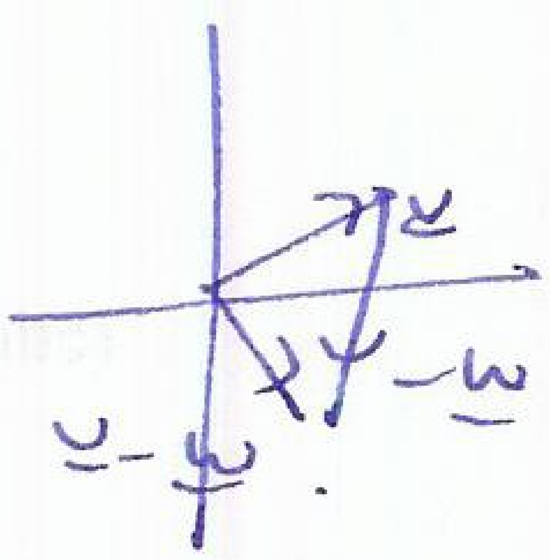




- $\underline{v} - \underline{w} = \underline{v} + (-1)\underline{w}$







vector operations using components:

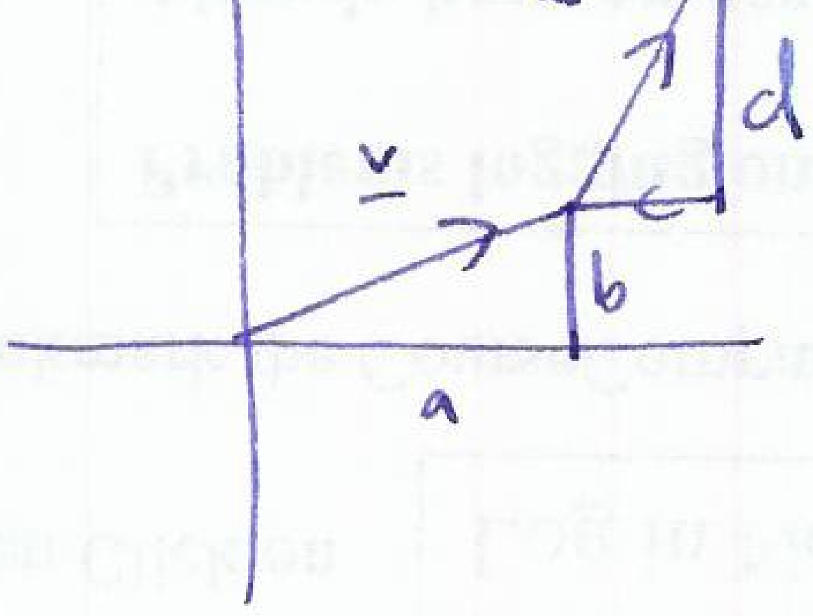
if $\underline{v} = \langle a,b \rangle$ and $\underline{w} = \langle c,d \rangle$

then
- $\underline{v}+\underline{w} = \langle a+c, b+d \rangle$
- $\underline{v}-\underline{w} = \langle a-c, b-d \rangle$
- $\lambda \underline{v} = \langle \lambda a, \lambda b \rangle$
- $\underline{v}+\underline{0} = \underline{0}+\underline{v} = \underline{v}$ and $\underline{v}-\underline{v} = \underline{0}$

important $\underline{v} - \underline{v} = \underline{0}$ zero vector $\langle 0, 0 \rangle$

<u>not</u> zero number 0.

$\underline{v} + \underline{w}$:



$\underline{v} + \underline{w} = \langle a+c, b+d \rangle$.

Useful properties $\underline{u}, \underline{v}, \underline{w}$ vectors, λ scalar.

- commutative $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
- associative $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$
- distributive $\lambda(\underline{v} + \underline{w}) = \lambda \underline{v} + \lambda \underline{w}$
- length $\|\lambda \underline{v}\| = |\lambda| \|\underline{v}\|$.

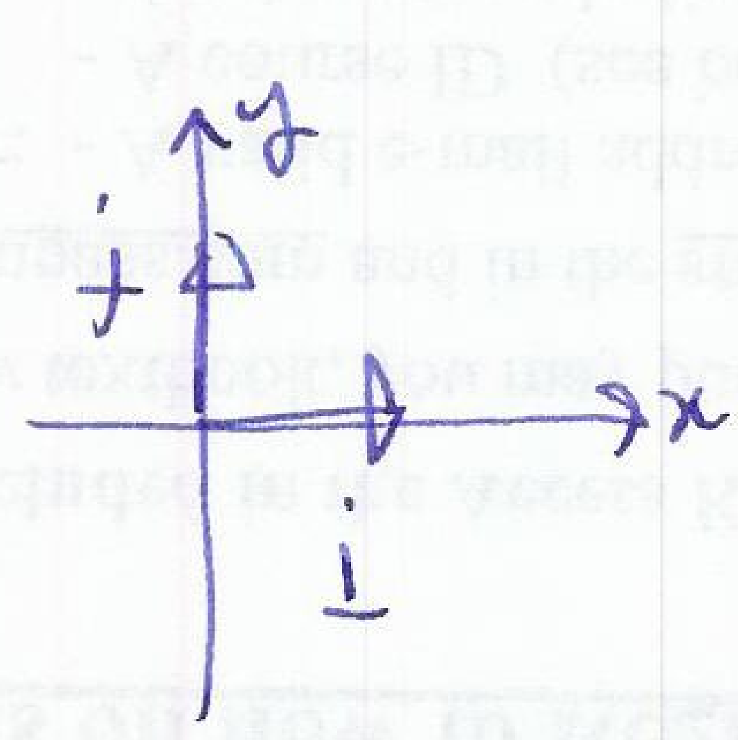
Important $\lambda + \underline{v}$ does not make sense!

Unit vectors: A vector of length 1 is called a unit vector

if $\underline{v}$ is a vector $(\neq \underline{0})$ then the vector $\underline{e}_v = \frac{1}{\|\underline{v}\|} \underline{v}$ is a unit

vector in the direction of $\underline{v}$. Check: $\|\underline{e}\| = \left\| \frac{1}{\|\underline{v}\|} \underline{v} \right\| = \frac{1}{\|\underline{v}\|} \|\underline{v}\| = 1$

Special vectors



$\underline{i}$ unit vector in x-direction

$\underline{j}$ unit vector in y-direction

so $\underline{i} = \langle 1, 0 \rangle$

$\underline{j} = \langle 0, 1 \rangle$

Linear combinations if $\underline{v}, \underline{w}$ are vectors, r, s scalars.

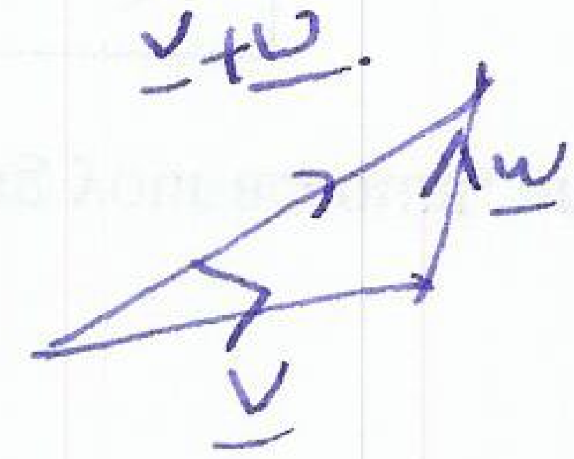
then $r\underline{v} + s\underline{w}$ is a <u>linear combination</u> of $\underline{v}$ and $\underline{w}$.

$\underline{i}, \underline{j}$ sometime called <u>standard basis vector</u>.

Every vector can be written as a linear combination of $\underline{i}$ and $\underline{j}$.

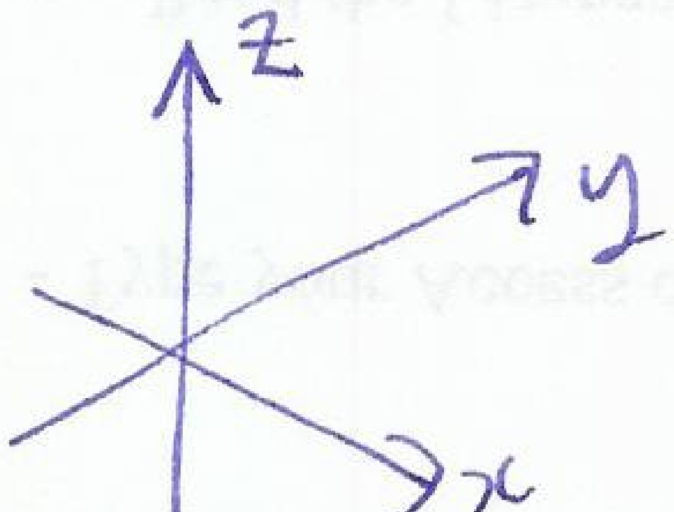
$$\underline{v} = \langle a, b \rangle = a\underline{i} + b\underline{j} = a\langle 1, 0\rangle + b\langle 0, 1\rangle$$
$$= \langle a, 0\rangle + \langle 0, b\rangle$$
$$= \langle a, b\rangle.$$

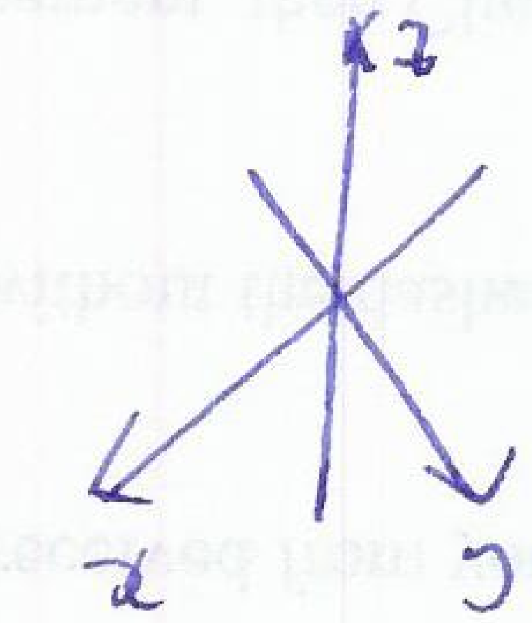
<u>Triangle inequality</u>



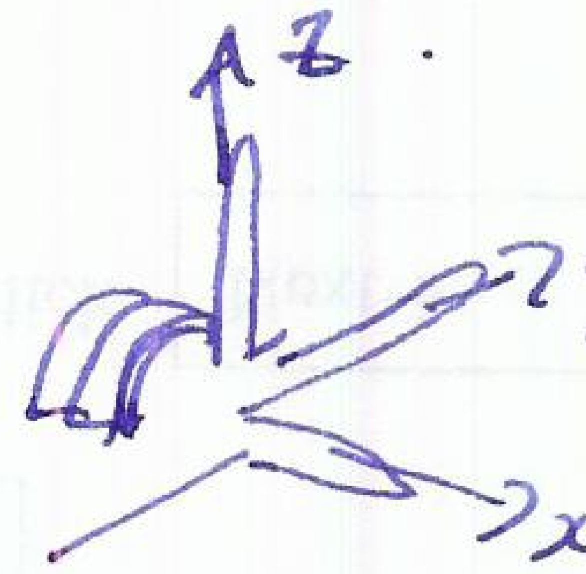
$$\|\underline{v}+\underline{w}\| \le \|\underline{v}\| + \|\underline{w}\|.$$

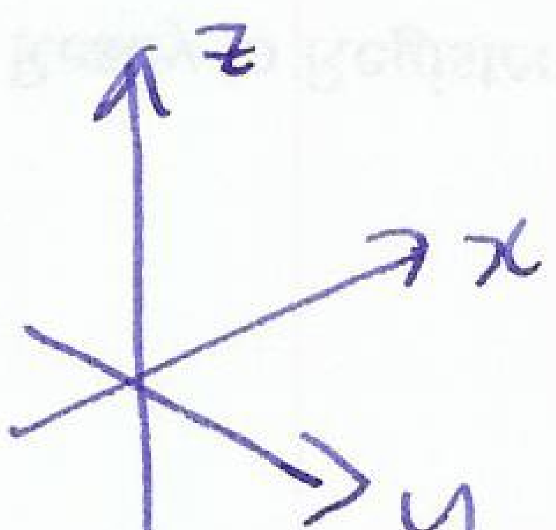
<u>§12.2 Vectors in 3d</u>





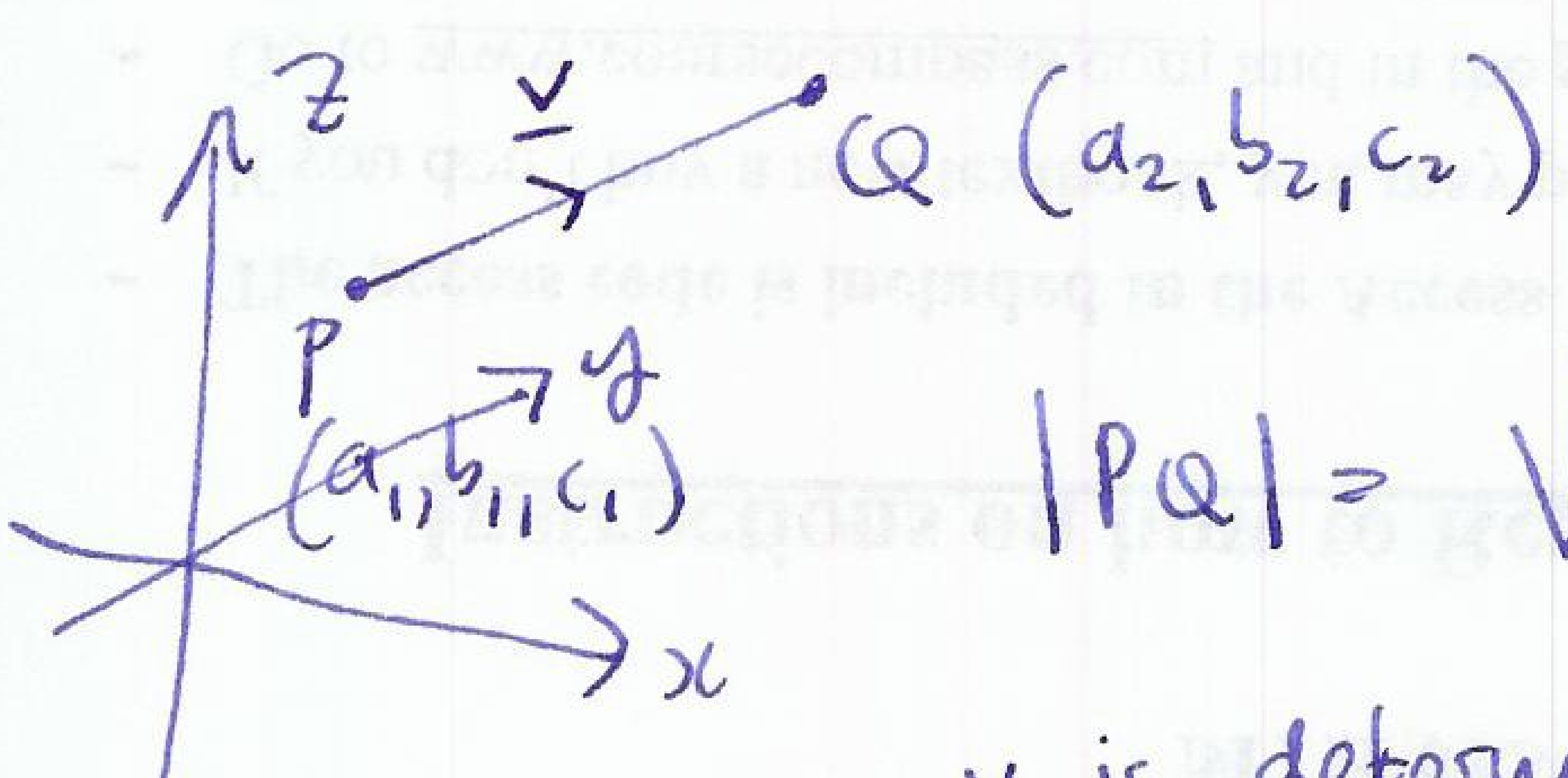
right hand rule ✓





left hand rule. ✗ we will only ever use coordinates which are right handed

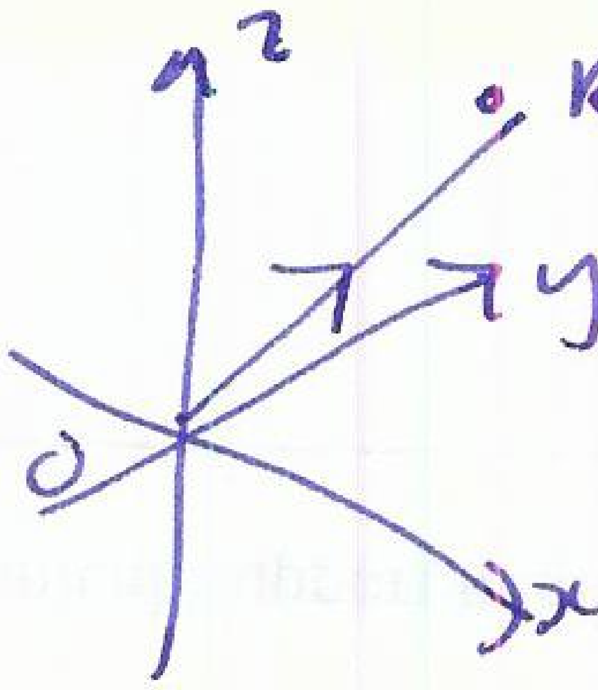
A vector $\underline{v}$ has length and direction.



$$|PQ| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2 + (c_2-c_1)^2}$$

$\underline{v}$ is determined by its initial and final points.

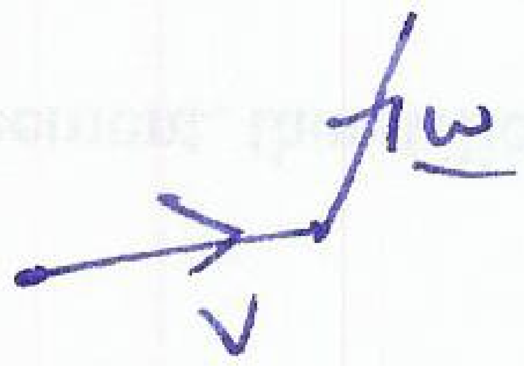
- translation: $\underline{v}$ moved without changing length or direction.
- equivalent: $\underline{v}, \underline{w}$ same length and direction.

• position vectors:  point  $\overrightarrow{OR}$ position vector.

• components $P\,(a_1, b_1, c_1)$, $Q\,(a_2, b_2, c_2)$ then components are

$a_2 - a_1$ x-component
$b_2 - b_1$ y-component
$c_2 - c_1$ z-component

notation $\overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$.

• vector addition 

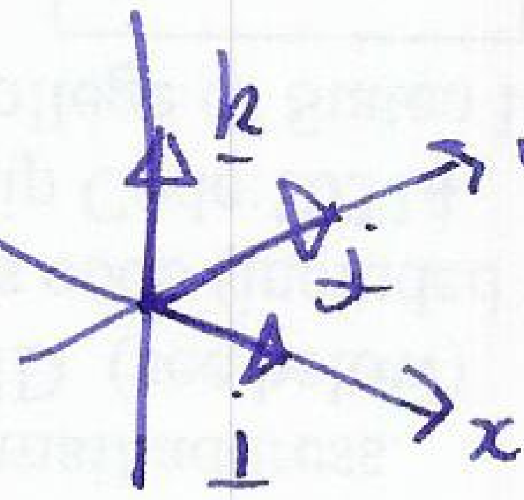
$\underline{v} = \langle v_1, v_2, v_3 \rangle$

$\underline{w} = \langle w_1, w_2, w_3 \rangle$

$\underline{v} + \underline{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$

• scalar multiplication

$\lambda \underline{v} = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$

• standard basis vectors 

$\underline{i} = \langle 1, 0, 0 \rangle$

$\underline{j} = \langle 0, 1, 0 \rangle$

$\underline{k} = \langle 0, 0, 1 \rangle$

every vector is a linear combination of the standard basis vectors.

$\underline{v} = \langle a, b, c \rangle = a\underline{i} + b\underline{j} + c\underline{k}$

$= a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle$