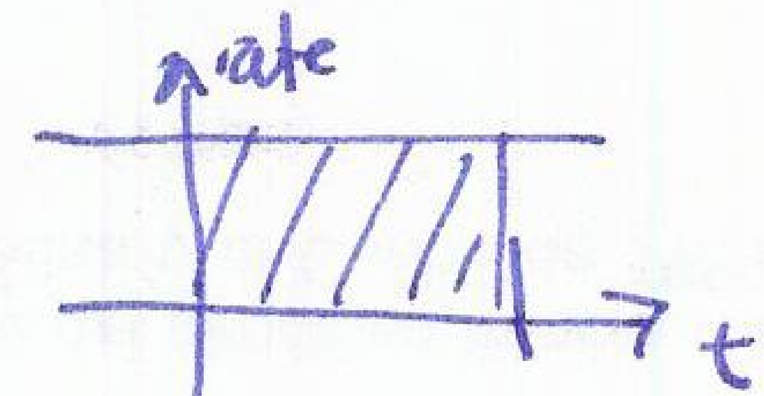
§5.5 Net change / applications of integrals

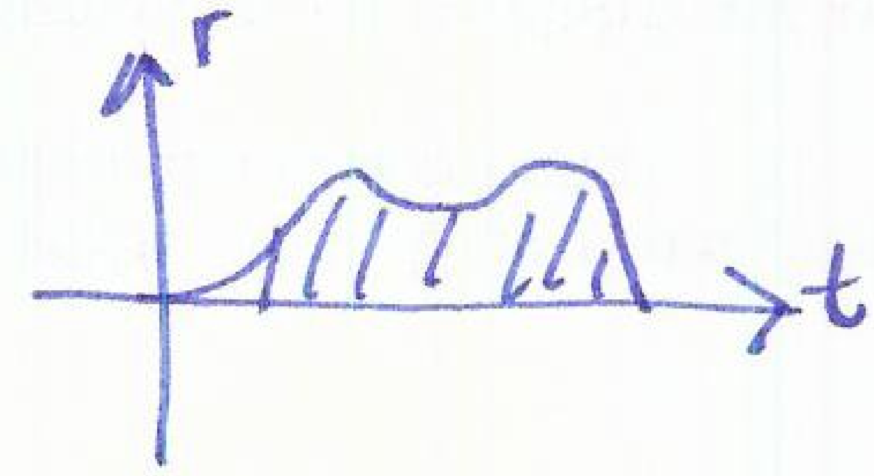
Example you pour water into a bucket at rate $r(t)$. How much water is in the bucket?

constant rate:



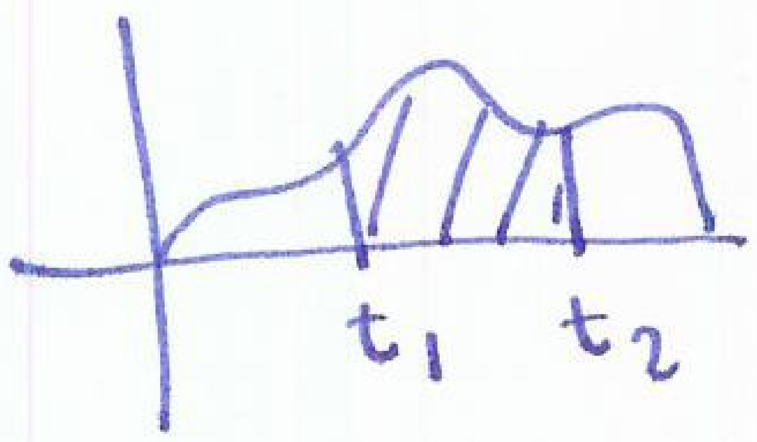
amount = rate × time.

varying rate $r(t)$: amount = area under curve



$$= \int_0^t r(x)\,dx$$

Q: by how much did the amount of water change between t_1 and t_2?



$$(\text{net change}) = \int_{t_1}^{t_2} r(x)\,dx$$

§5.6 Substitution / change of variable

"reverse chain rule for integration"

recall: chain rule: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$

substitution / change of variable:

$\int f(u)\,du \quad u(x)$

$$\boxed{\int_a^b f(u)\,du = \int_{u^{-1}(a)}^{u^{-1}(b)} f(u(x))\,\frac{du}{dx}\,dx}$$

remember: three things to change: limits, function, differential

mnemonic: $du = \frac{du}{dx}\,dx$ "cancelling fractions".

why does this work?

$\int f(u)\,du \quad u(x) \quad$ set $F(u) = \int f(u)\,du$ so $F'(u) = f(u)$

$$\int f(u)\,du = F(u) \underset{\text{wrt } x}{\overset{\text{differentiate}}{\rightsquigarrow}} \frac{d}{dx}\left(F(u(x))\right) = F'(u(x))\,u'(x) = f(u(x))\,u'(x)$$

$$\underset{\text{wrt } x}{\overset{\text{integrate}}{\rightsquigarrow}} : \int f(u(x))\,u'(x)\,dx = \int f(u(x))\,\frac{du}{dx}\,dx \quad \square$$

Examples $\int_{x=0}^{x=1} e^{-7x}\,dx$ set $u=-7x$ useful fact: $\frac{dx}{du} = \frac{1}{\frac{du}{dx}}$.

$\frac{du}{dx} = -7$

$$\int_{u=0}^{u=-7} e^u \frac{dx}{du}\,du = \int_0^{-7} e^u \cdot -\frac{1}{7}\,du = -\frac{1}{7}\left[e^u\right]_0^{-7} = -\frac{1}{7}\left(e^{-7}-1\right)$$

more examples: $\int_0^2 x^2\sqrt{x^3+1}\,dx$, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$ $\int_0^1 \frac{x}{x^2+1}\,dx$

§5.7 More integrals

recall $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ so $\int \frac{1}{x}\,dx = \ln|x| + C$

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ so $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$

$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{x^2+1}$ $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$

$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{|x|\sqrt{x^2-1}}$ $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + C$

$\frac{d}{dx}(e^x) = e^x$ $\int e^x\,dx = e^x + C$

recall: $b^x = e^{x\ln(b)}$ $\int b^x\,dx = \int e^{x\ln(b)}\,dx = \frac{1}{\ln(b)} e^{x\ln(b)} + C$

(sub $u = x\ln(b)$).

Example. $\int \frac{1}{9+x^2}\,dx = \frac{1}{9}\int \frac{1}{1+x^2/9}\,dx = \frac{1}{9}\int \frac{1}{1+(x/3)^2}\,dx$

substitute $u = x/3$ ....

- $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}\,dx$ substitute $u = 2x$...

Examples

net change:

Q1 water flows in to the reservoir at a rate of $3000 + 5t$ gal/hour

how much water flows in the 4th hour? $\int_3^4 3000 + 5t\,dt$

$$= \left[3000t + \frac{5}{2}t^2\right]_3^4 = 3000 + \frac{5}{2}(16-9) = 3000 + \frac{35}{2} \text{ gallons.}$$

substitution $\int x(x+1)^9\,dx \qquad \int \sqrt{4x-1}\,dx$

$$\int \sin^2 x \cos x\,dx \qquad \int x\cos(x^2)\,dx \qquad \int x\sqrt{x^2-1}\,dx$$

$$\int \frac{1}{x\ln x}\,dx$$

other $\int 2^x\,dx \qquad \int \frac{e^{2x}+e^{4x}}{e^x}\,dx$

Final : Thu 17th May 1:30 - 3:30pm 15-219

ww : Wed 16th

sample final : Spring 2011.