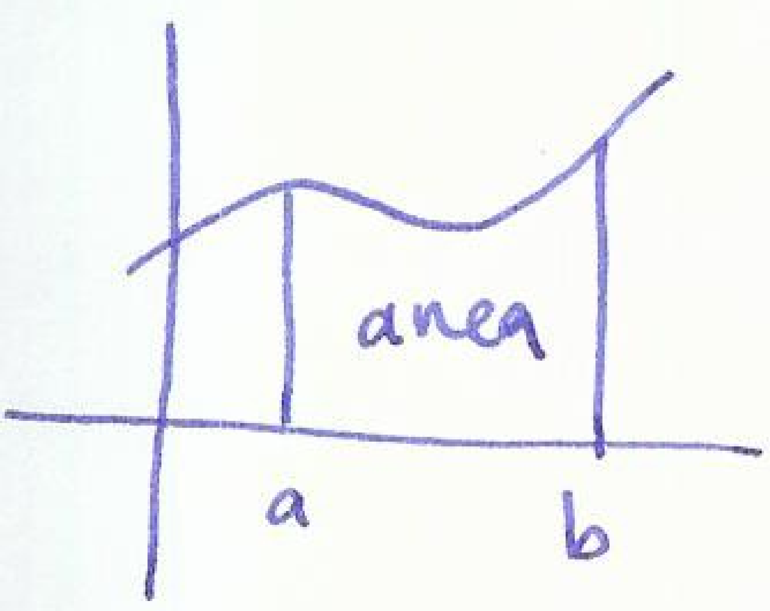
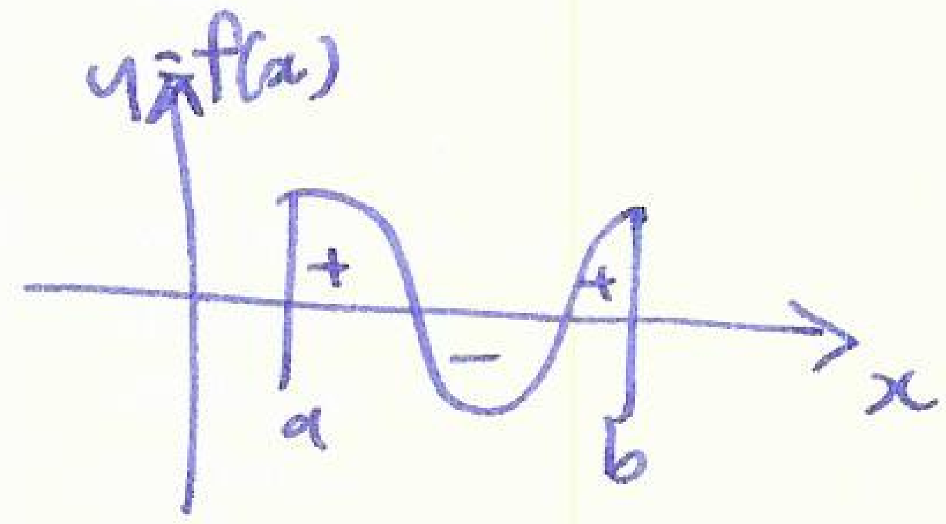
§5.2 Definite integral

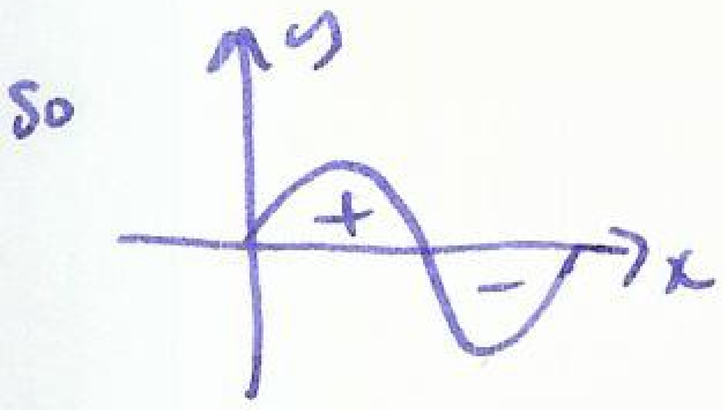


$\int_a^b f(x)\,dx$ = area under curve $y=f(x)$ between $x=a$ and $x=b$

note: signed area:

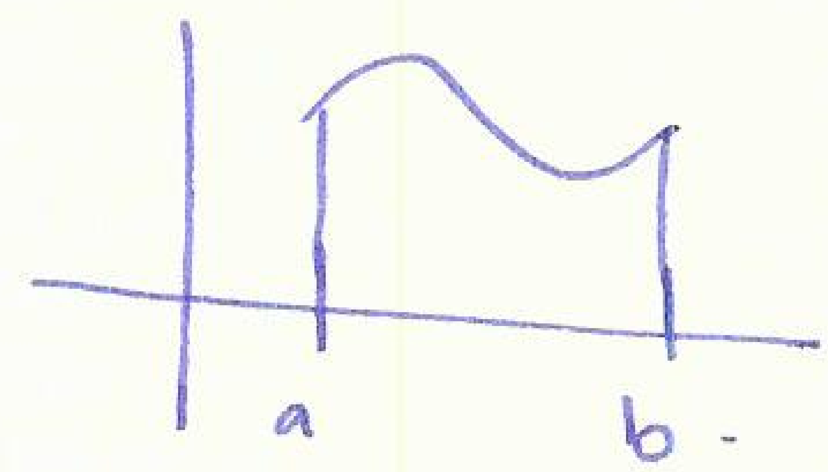


so



$\int_0^{2\pi} \sin(x)\,dx = 0$

Formal definition: Riemann sum $R(f,P,c)$



P = partition of $[a,b]$

$a = x_0 \le x_1 \le x_2 \cdots \le b = x_n$

c = choice of points $c_i \in [x_{i-1}, x_i]$ width $\Delta x_i = x_i - x_{i-1}$

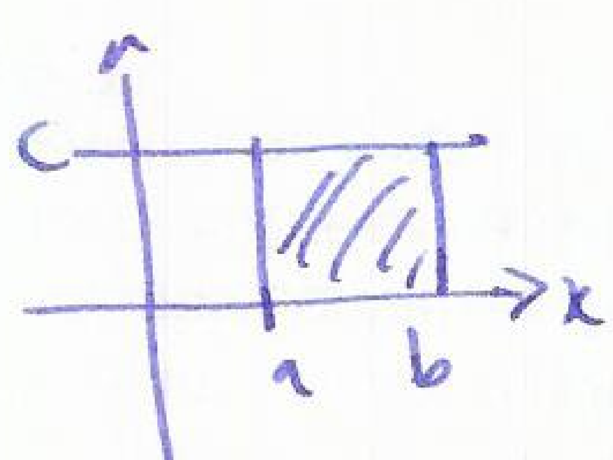
$R(f,P,c) = \sum_{i=1}^{n} f(c_i)\Delta x_i$ $\qquad ||P|| = \max \Delta x_i$

Defn $\int_a^b f(x)\,dx = \lim_{||P|| \to 0} R(f,P,c) = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$

When this limit exists we say f is integrable over $[a,b]$

Useful properties

$\int_a^b c\,dx = c(b-a)$



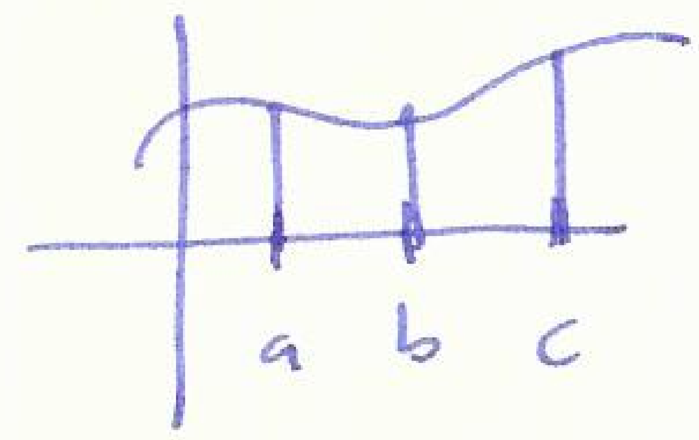
$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

Reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

0-length integral: $\int_a^a f(x)\,dx = 0$

adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

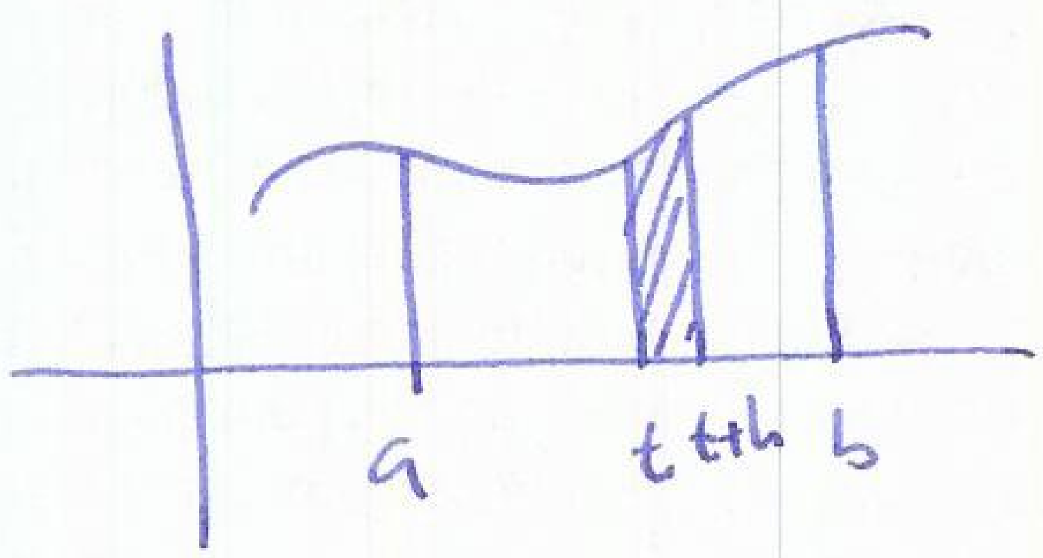


comparisons: if $f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

§5.3 Fundamental theorem of calculus

Thm (FTC ①) Suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an anti-derivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

intuition:



consider $\int_a^t f(x)\,dx$ ← rate of change wrt t

↔ adding an small rectangle of area $\approx f(t).h$

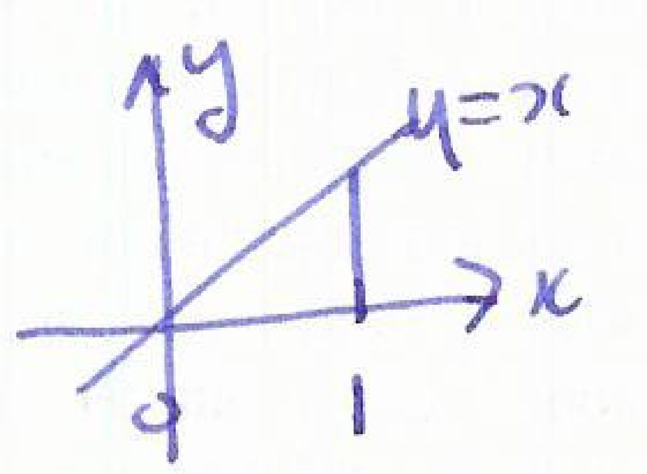
i.e. $\lim_{h\to 0} \frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} \approx \lim_{h\to 0} \frac{f(t)h}{h} = f(t)$

so $\frac{d}{dt}\left(\int_a^t f(x)\,dx\right) = f(t)$.

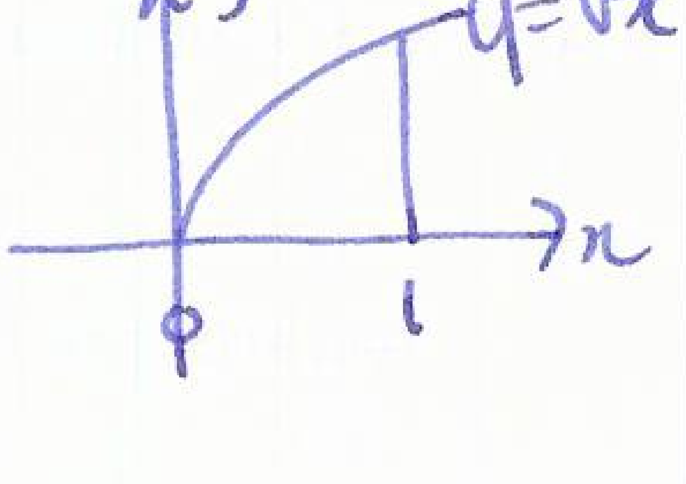
i.e. $\int_a^t f(x)\,dx$ is an antiderivative for $f(x)$ so $\int_a^t f(x)\,dx = F(t) + c$

what is the constant? $t=a$: $\int_a^a f(x)\,dx = F(a) + c = 0$ so $c = -F(a)$.

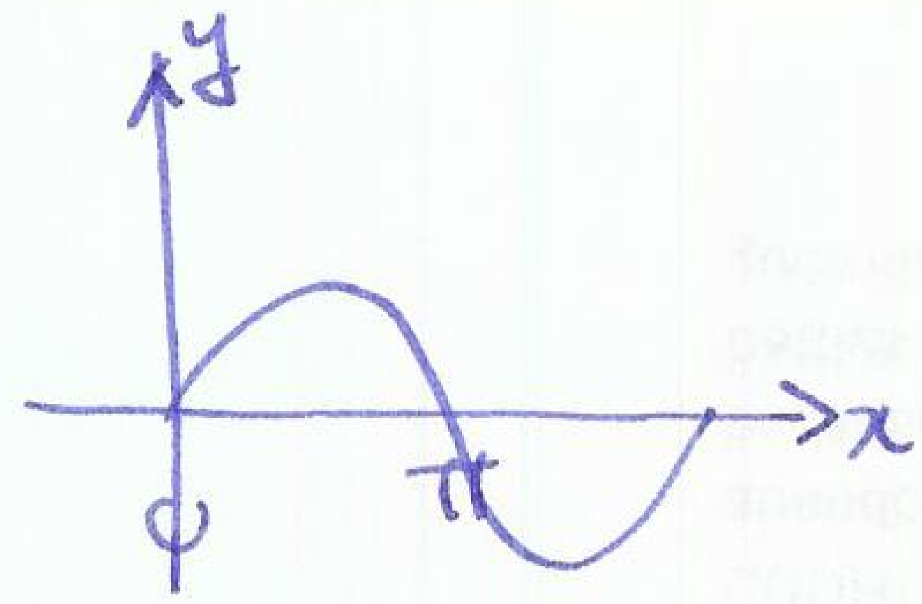
Examples



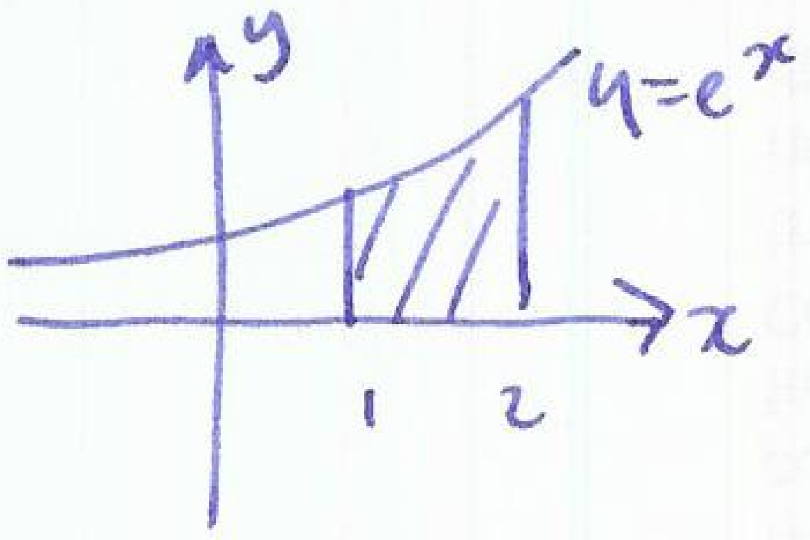
$\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$.



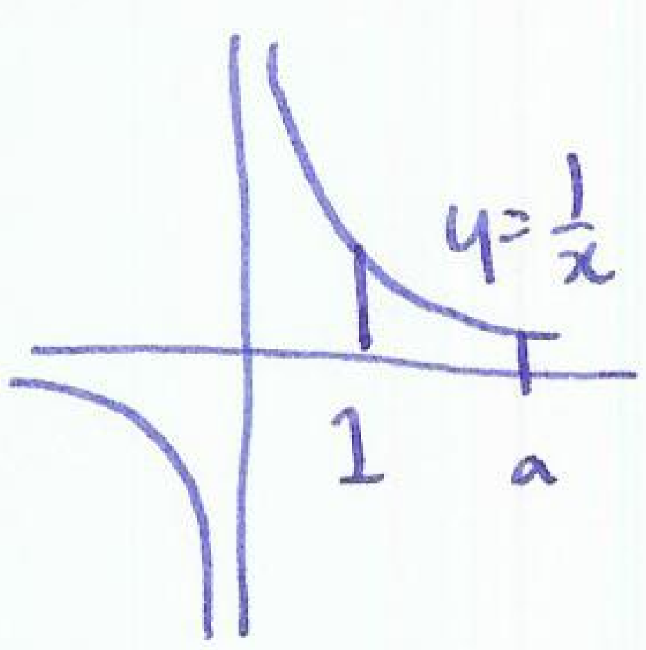
$\int_0^1 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$.



$$\int_0^{\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi} = -\cos(\pi) + \cos(0) = -(-1)+1 = 2.$$



$$\int_1^2 e^x\,dx = \Big[e^x\Big]_1^2 = e^2 - e$$



$$\int_1^a \frac{1}{x}\,dx = \Big[\ln x\Big]_1^a = \ln(a) - 0 = \ln(a) \to \infty \text{ as } a \to \infty.$$