Notation: indefinite integral

$$\int f(x)\,dx = F(x) + c \quad \text{means: } F(x)+c \text{ is the general antiderivative for } f(x)$$

Thm $\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c$ for $n \neq -1$

Proof $\frac{d}{dx}\left(\frac{1}{n+1} x^{n+1} + c\right) = \frac{1}{n+1}\, n+1\, x^n = x^n$ $\square$

Thm $\int \frac{1}{x}\,dx = \ln|x| + c$

Proof $(x>0)$ $\frac{d}{dx}\left(\ln(x) + c\right) = \frac{1}{x}$ $\square$

Thm sums and constant multiples:

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Warning no product/quotient/chain rule!

Useful integrals

$$\int \sin x\,dx = -\cos x + c \qquad \int e^x\,dx = e^x + c$$

$$\int \cos x\,dx = \sin x + c$$

Example $\int x^2 + \frac{1}{x} + \sin x\,dx = \frac{1}{3}x^3 + \ln|x| \cancel{+} - \cos x + c$.

Alternate view

We can think of finding the derivative as solving a differential equation $\frac{dy}{dx} = f(x)$. In general there is a family of solutions $F(x)+c$, but if we know the value of the solution we want at $x=0$ (sometimes called an initial condition) this gives a particular solution.

Example an object falls freely under gravity, so

acceleration $a(t) = -g$ (constant)

velocity $v(t)$, so $v'(t) = a(t)$

$\frac{dv}{dt} = -g$ has general solution $v = -gt + c$

if the velocity at time $t=0$ is v_0, then $v_0 = -g \cdot 0 + c \Rightarrow c = v_0$

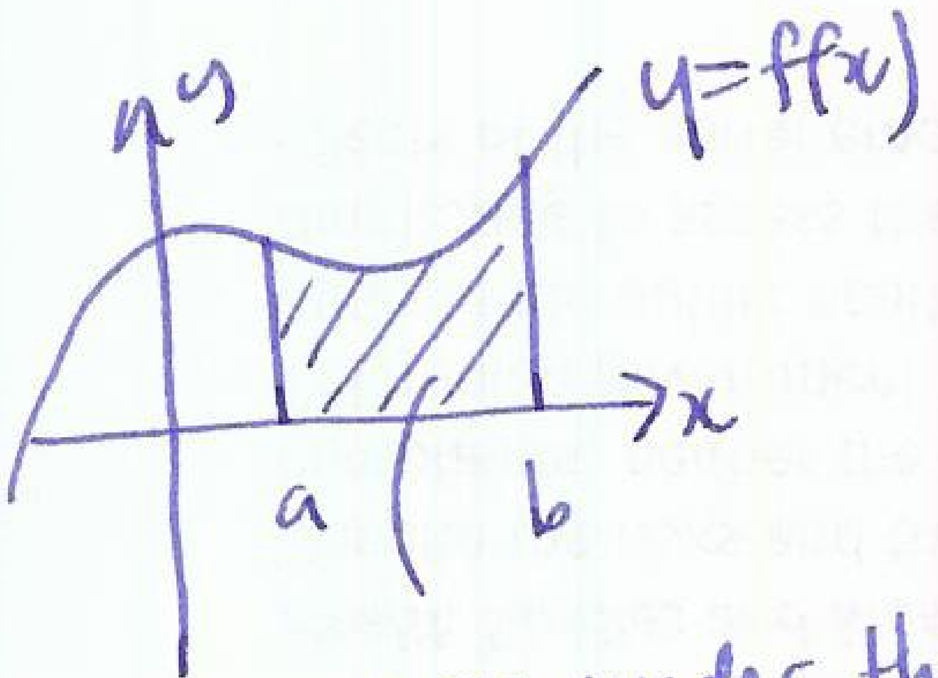
so the particular solution is $v = -gt + v_0$.

position $x(t)$, so $\frac{dx}{dt} = v(t)$

$\frac{dx}{dt} = -gt + v_0$ has general solution $x = -\frac{1}{2}gt^2 + v_0 t + c$

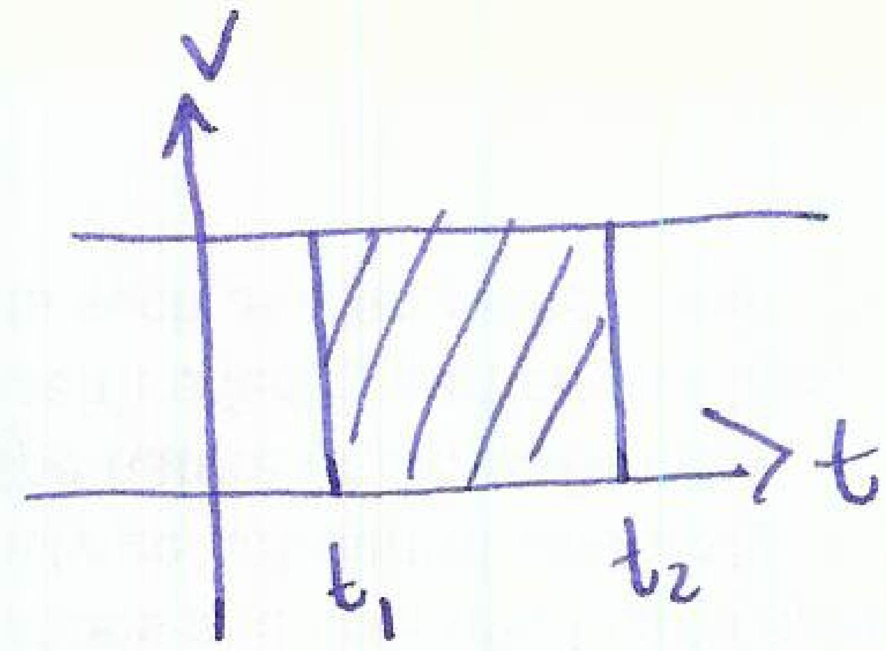
if the position at time $t=0$ is x_0, then $x = -\frac{1}{2}gt^2 + v_0 t + x_0$.

§5.1 Approximating areas



area under the graph

example:

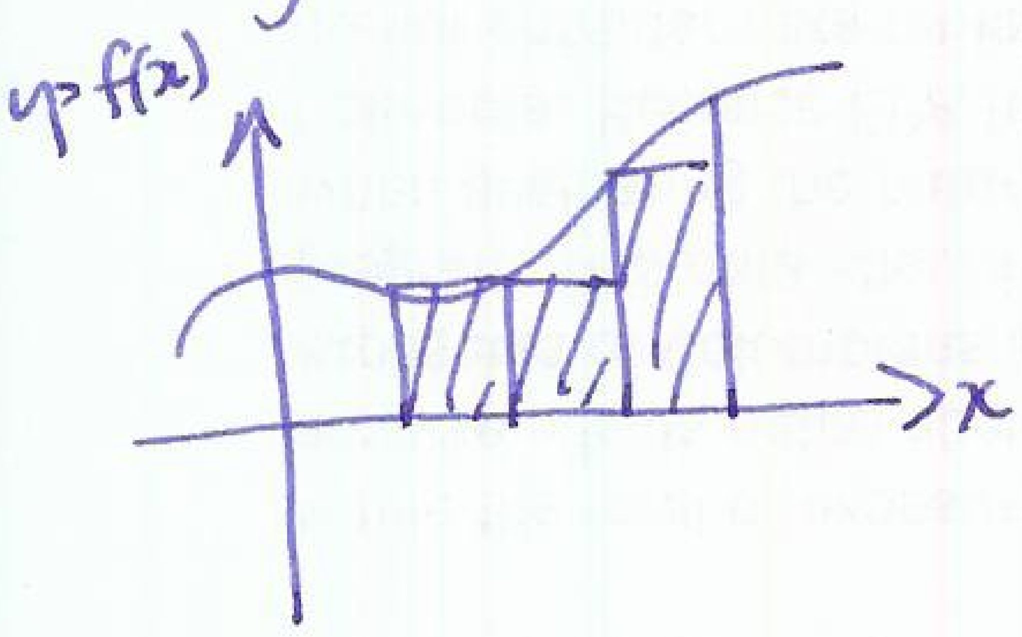


constant velocity

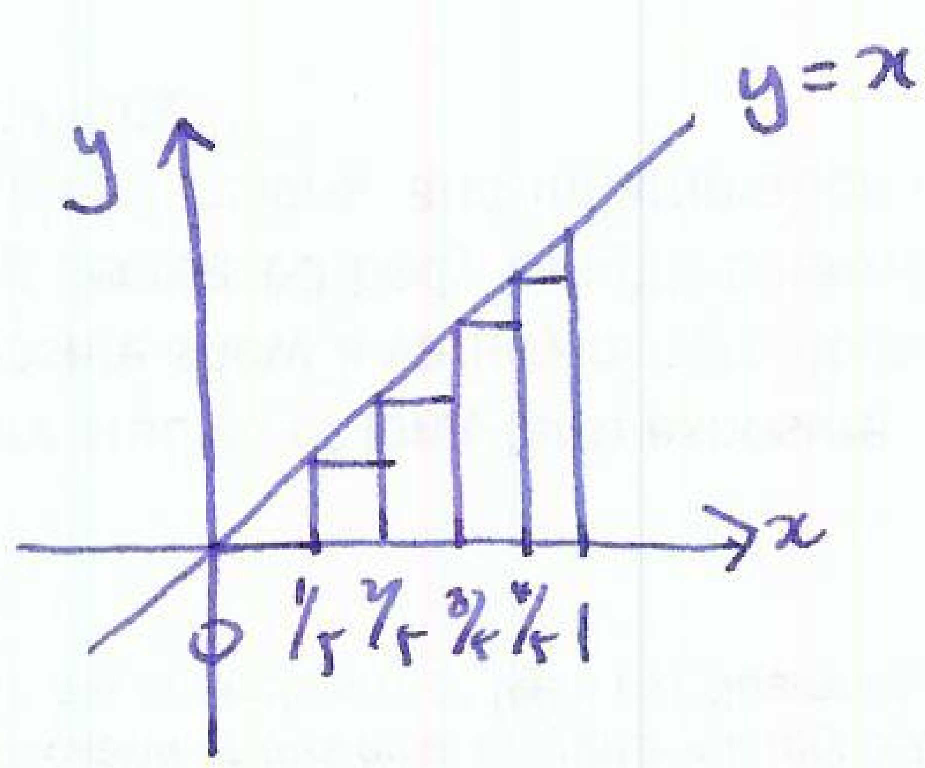
distance travelled = velocity × time = area under velocity graph.

area = distance travelled.

finding the area: approximate by rectangles.



Example



find area under $y=x$ between 0 and 1 (answer is $\frac{1}{2}$)

approximate with 5 rectangles: area $\approx \sum$ rectangles $= \sum f(x_i)\,$width.

$$= \sum_{i=0}^{4} f\left(\tfrac{i}{5}\right)\cdot\frac{1}{5} = \frac{1}{5}\left(f(0)+f\left(\tfrac{1}{5}\right)+f\left(\tfrac{2}{5}\right)+f\left(\tfrac{3}{5}\right)+f\left(\tfrac{4}{5}\right)\right).$$

$$= \frac{1}{5}\left(0+\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right) = \frac{1}{25}\cdot 10 = \frac{10}{25}.$$

n rectangles: $\displaystyle\sum_{i=0}^{n-1} f\left(\tfrac{i}{n}\right)\frac{1}{n} = f(0)\frac{1}{n} + f(1)\frac{1}{n} + \cdots + f(n-1)\frac{1}{n}.$

$$= \frac{1}{n}\left(0+\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}\right) = \frac{1}{n^2}\left(1+2+\cdots+n-1\right)$$

$$= \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

claim: $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$

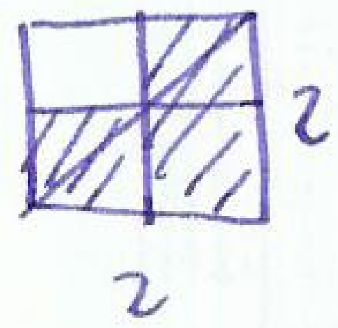
Proof ① induction: assume true for k:

$$s_n = 1+2+3+\cdots+k = \frac{1}{2}k(k+1)$$

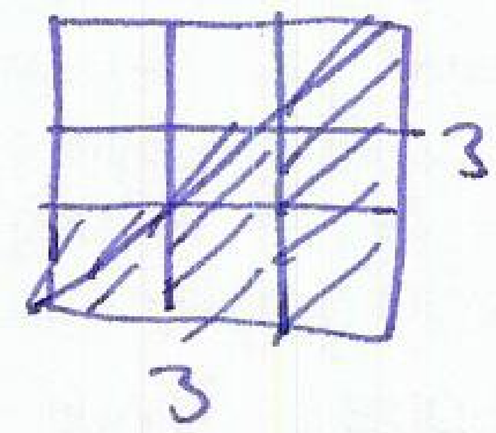
$$s_{n+1} = 1+2+3+\cdots+k+k+1 = \frac{1}{2}k(k+1)+k+1$$

$$= (k+1)\left(\tfrac{1}{2}k+1\right) = \tfrac{1}{2}(k+1)(k+2) \checkmark$$

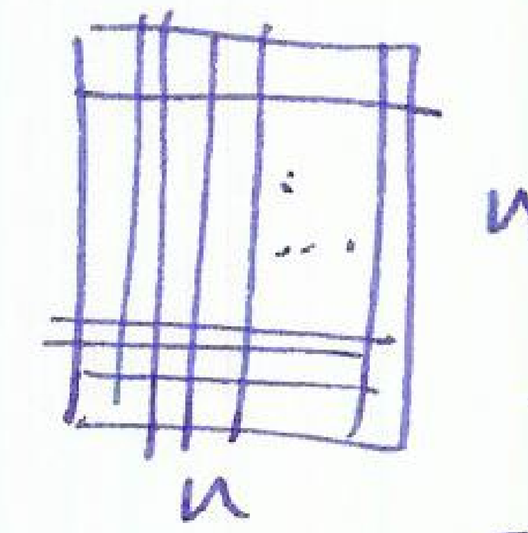
②



$\frac{1}{2}(2)^2+\frac{1}{2}(2)$



$\frac{1}{2}(3)^2+\frac{1}{2}(3)$

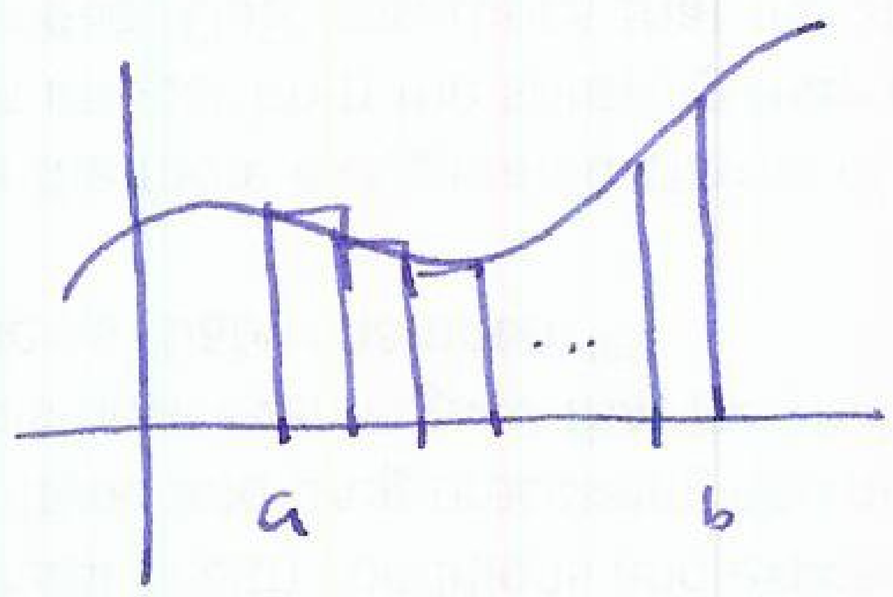


$\frac{1}{2}n^2+\frac{1}{2}n$

$=\frac{1}{2}n(n+1)$ D

so approximate area is $\frac{1}{n^2}\left(\frac{1}{2}(n-1)n\right) = \frac{n-1}{2n} = \frac{1}{2} - 1/2n \to \frac{1}{2}$ as $n\to\infty$.

Notation



suppose there are N rectangles of equal width, then $\Delta x = \frac{b-a}{N}$

then

left endpoint rectangles $\quad L_N = \sum_{j=0}^{N-1} f(a+j\Delta x)\Delta x$

right endpoint rectangles $\quad R_N = \sum_{j=1}^{N} f(a+j\Delta x)\Delta x$

midpoint rectangles $\quad M_N = \sum_{j=1}^{N} f(a+(j-\frac{1}{2})\Delta x)\Delta x$

useful fact:

Thm If $f(x)$ is continuous on $[a,b]$ then all three approximations give the same area under the curve:

$$\text{area} = \lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \lim_{N\to\infty} M_N$$