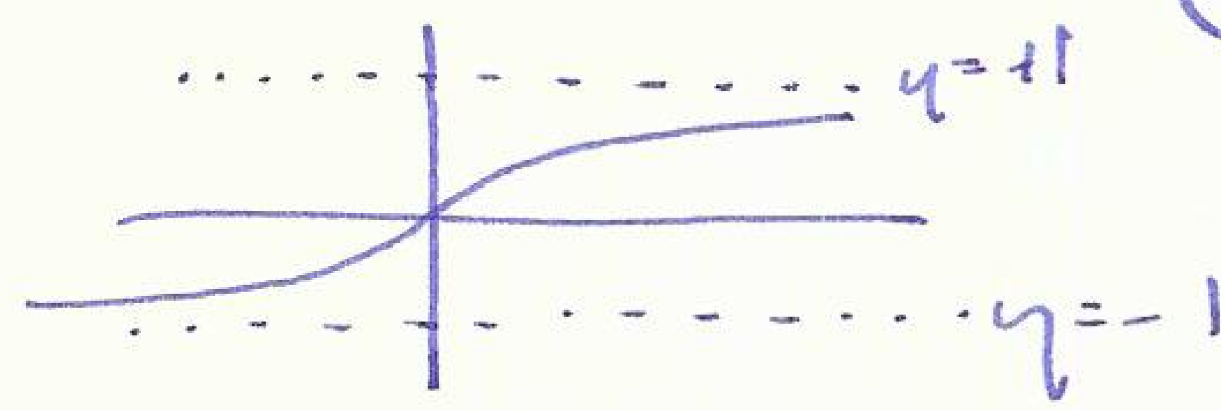
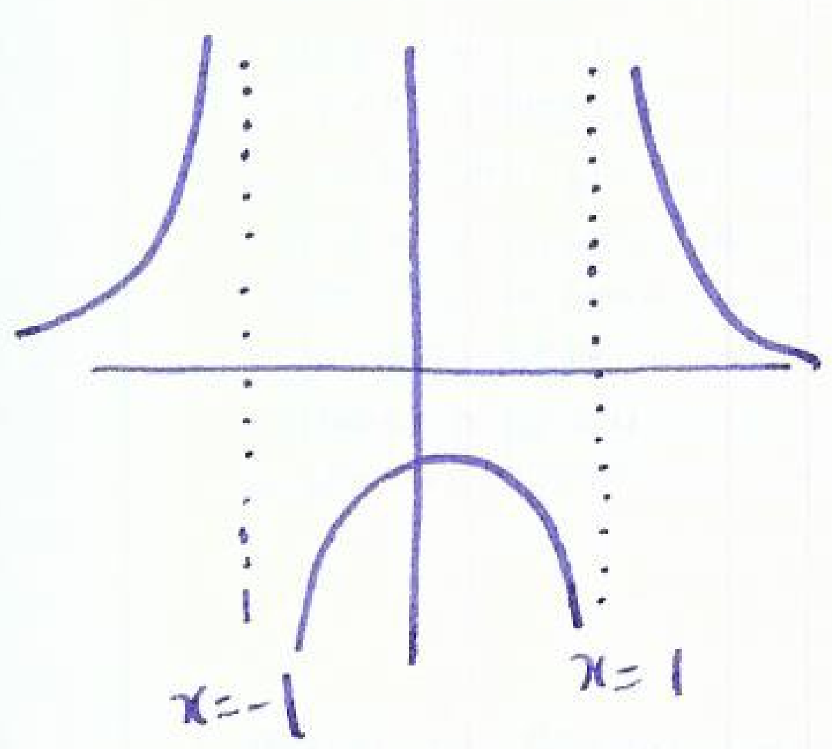
$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to \infty} \frac{-x}{\sqrt{x^2+1}} = -1.$$



Example $f(x) = \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$



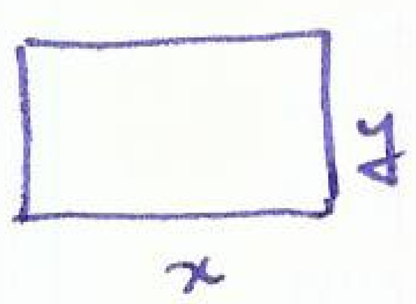
① vertical asymptotes at $x = \pm 1$

② $f'(x) = -(x^2-1)^{-2} \cdot 2x$ critical point at $x=0$.

③ $f''(x) = \frac{6x^2+2}{(x^2-1)^3}$. etc.

§4.6 Optimization

Example A piece of wire of length L is bent into a rectangle. What's the largest area rectangle possible?



area $A = xy$

perimeter $L = 2x + 2y \Rightarrow y = \frac{L}{2} - x$ so $A = x\left(\frac{L}{2} - x\right)$

$= \frac{L}{2}x - x^2$.

$A'(x) = \frac{L}{2} - 2x$ critical point $x = \frac{L}{4}$. (max)

so max area occurs at $x = y = \frac{L}{4}$ (ie a square)

Example What shape of cylindrical can minimizes surface area, if you want total volume to be 1 ft^3?

$V = \pi r^2 h = 1 \iff h = \frac{1}{\pi r^2}$

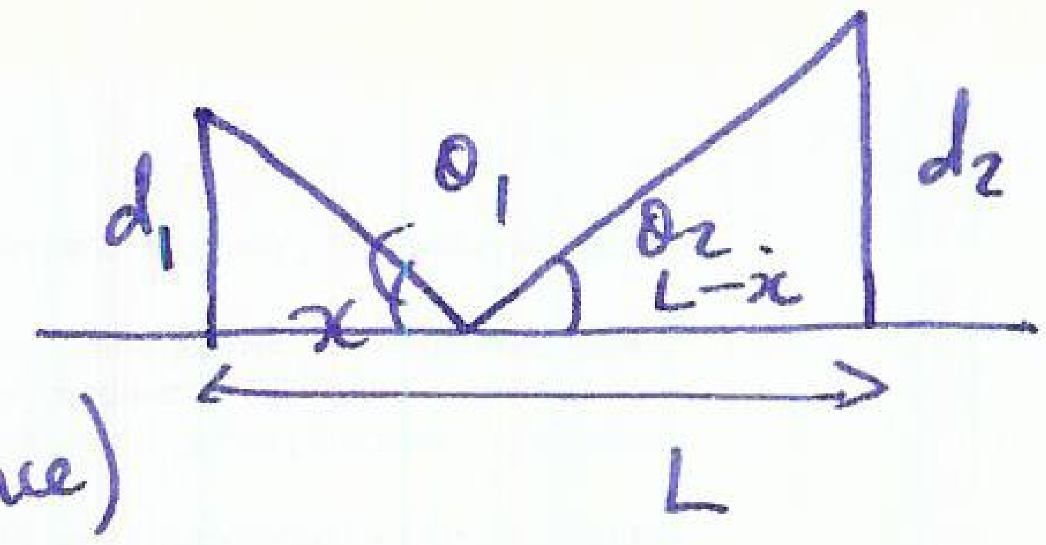
$A = 2\pi r^2 + 2\pi r h$

$A(r) = 2\pi r^2 + \frac{2\pi r \cancel{h}}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$

$A'(r) = 4\pi r - \frac{2}{r^2} = 0 \Rightarrow r^3 = \frac{1}{2\pi}$ $r = \sqrt[3]{\frac{1}{2\pi}}$.

Example a ball bounces off a wall.

show that the path which minimizes distance has equal angles. (angle of reflection = angle of incidence)



$$f(x) = \text{distance} = \sqrt{d_1^2+x^2} + \sqrt{d_2^2+(L-x)^2}.$$

$$f'(x) = \frac{1}{2}\left(d_1^2+x^2\right)^{-1/2}.2x + \frac{1}{2}\left(d_2^2+(L-x)^2\right)^{-1/2}.\,2(L-x).-1 = 0.$$

i.e.

$$\underbrace{\frac{x}{\sqrt{d_1^2+x^2}}}_{\cos\theta_1} = \underbrace{\frac{L-x}{\sqrt{d_2^2+(L-x)^2}}}_{\cos\theta_2} \Rightarrow \theta_1 = \theta_2.$$

§4.7 L'Hôpital's rule

Thm suppose $f(x)$ and $g(x)$ are differentiable and

$f(a) = g(a) = 0$ (or $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$)

and $g'(x) \neq 0$ near a. Then $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

(provided this limit exists)

② $f(a)/g(a)$ must be indeterminate form.

Warning ① this is nothing to do with the quotient rule!

Examples

① $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{3x^2}{1} = 3.$

② $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\cos^2 x}{\sin x - 1} = \lim_{x\to\frac{\pi}{2}}\frac{-2\cos x\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}} -2\sin x = -2$

③ $\displaystyle\lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{\ln x}{1/x} = \lim_{x\to 0+}\frac{1/x}{-x^{-2}} = \lim_{x\to 0+} -x = 0$

④ $\displaystyle\lim_{x\to 0}\frac{e^x-x-1}{\cos x-1} = \lim_{x\to 0}\frac{e^x-1}{-\sin x} = \lim_{x\to 0}\frac{e^x}{-\cos x} = -e$

(5) $$\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = 0$$

(6) $$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\ln x} \quad : \text{note } e^x \text{ continuous so } = e^{\left(\lim_{x\to 0+} x\ln x\right)}$$

$$\lim_{x\to 0+} x\ln x = 0 \quad \text{so} \quad \lim_{x\to 0+} x^x = e^0 = 1$$

Comparing growth rates of functions

Q: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln(x))^2} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{2\ln(x)\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{4/x}$$

$$= \lim_{x\to\infty} \frac{1}{8}x^{1/2} = \infty. \quad \text{so } \sqrt{x} \text{ grows faster.}$$

Thm e^x grows faster than any polynomial $p(x)$.

Proof $$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty.$$