

§4.5 Graph sketching and asymptotes

(56)

Example sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$

helpful info:

- critical points
- sign of f'
- sign of f''

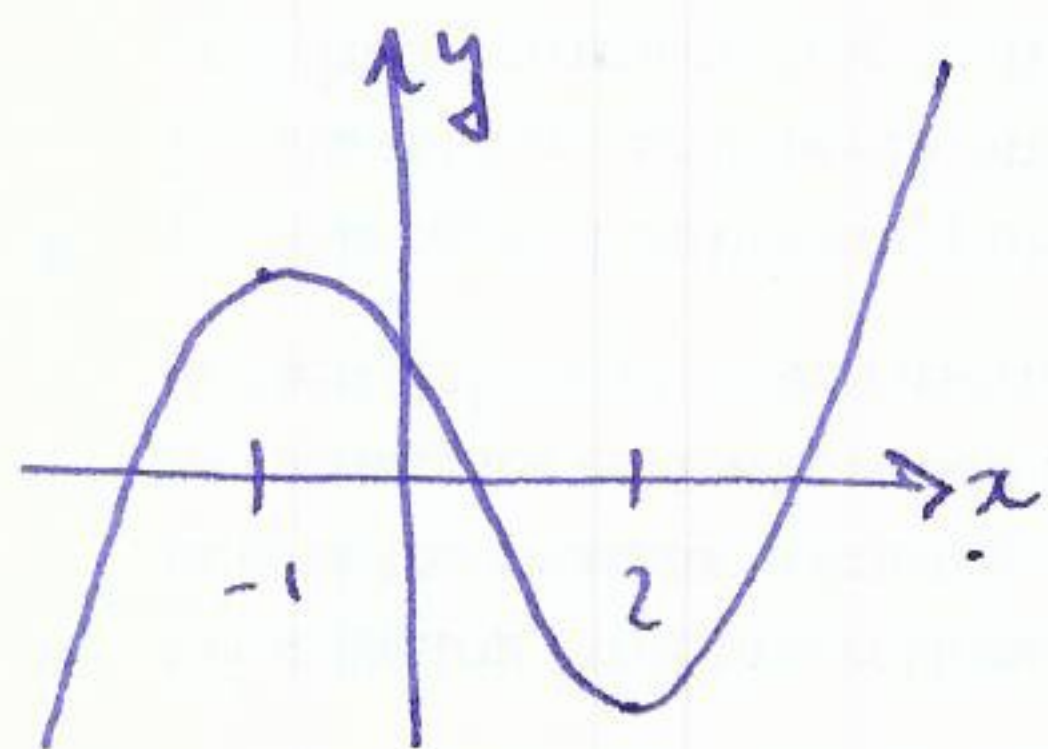
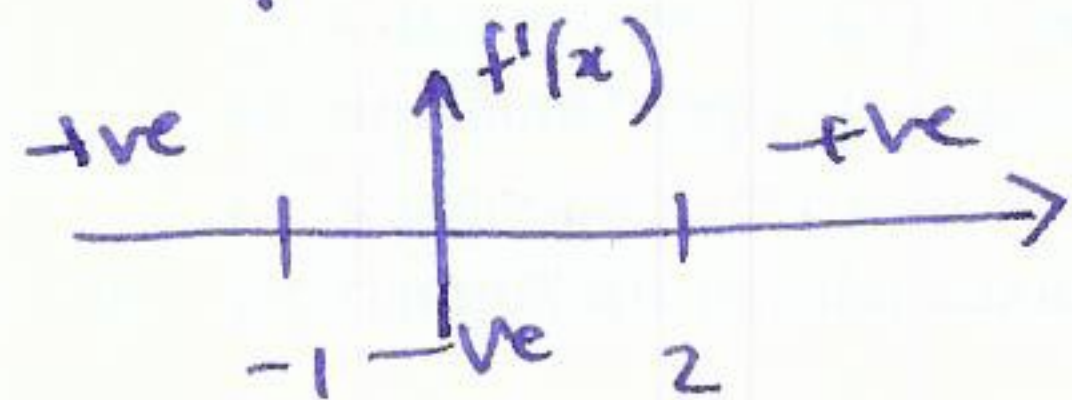
$$f'(x) = x^2 - x - 2 = (x+1)(x-2)$$

$$f''(x) = 2x - 1$$

critical points at $x = -1, 2$

$$f''(-1) = -3 < 0 \text{ max}$$

$$f''(2) = 3 > 0 \text{ min}$$

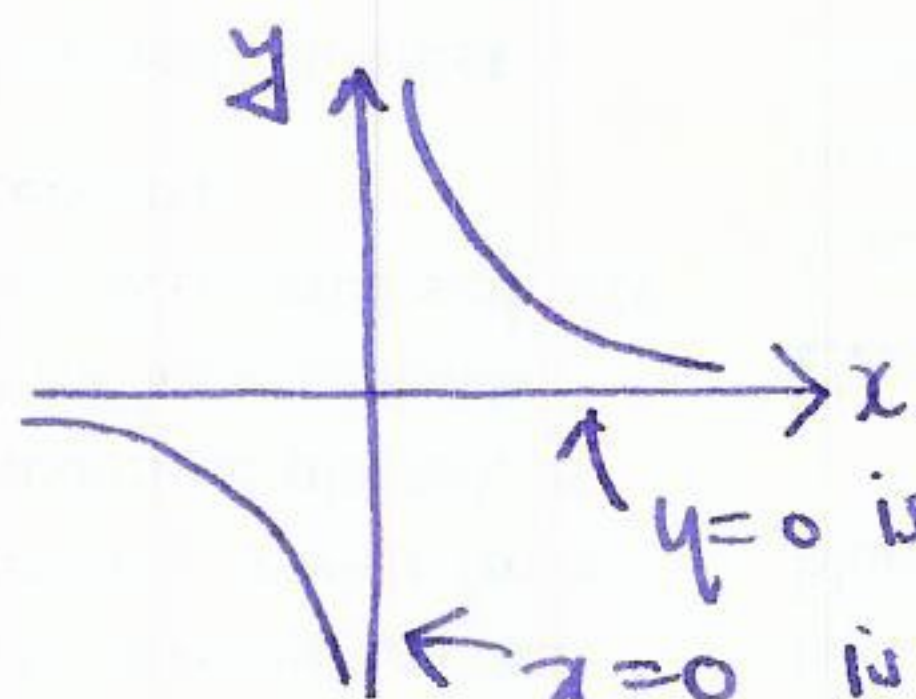


$$f(-1) = -\frac{1}{3} + \frac{1}{2} + 2 + 3 > 0$$

$$f(2) = \frac{8}{3} - 2 - 4 + 3 < 0$$

Asymptotes

Example $y = \frac{1}{x}$



$y=0$ is horizontal asymptote
 $x=0$ is vertical asymptote.

Defn $y=c$ is a vertical asymptote if $\lim_{x \rightarrow c^\pm} f(x) = \pm \infty$

$y=c$ is a horizontal asymptote if $\lim_{x \rightarrow \pm \infty} f(x) = c$

Observation rational functions have horizontal asymptotes if $\deg p < \deg q$

Example $f(x) = \frac{x^2 + x + 1}{3x^2 + 2} = \frac{1 + 1/x + 1/x^2}{3 + 2/x^2} \rightarrow \frac{1}{3}$ as $x \rightarrow \infty$.

Example find critical points of $f(x) = (4x - x^2)e^x$, and identify.

$$\begin{aligned} f'(x) &= 4xe^x + 4e^x - x^2e^x - 2xe^x = (4x + 4 - x^2 - 2x)e^x \\ &= (4 + 2x - x^2)e^x \end{aligned}$$

$$\text{solve } f'(x) = 0 \quad x = \frac{2 \pm \sqrt{4 + 16}}{2} = 1 \pm \sqrt{5}$$

find $f''(x) = (4+2x-x^2)e^x + (2-2x)e^x = (6-x^2)e^x$.

$f''(1+\sqrt{5}) < 0$ local max

$f''(1-\sqrt{5}) > 0$ local min

Example sketch graph of $\frac{3x+2}{2x-4}$

① find vertical asymptotes (when denominator is zero)

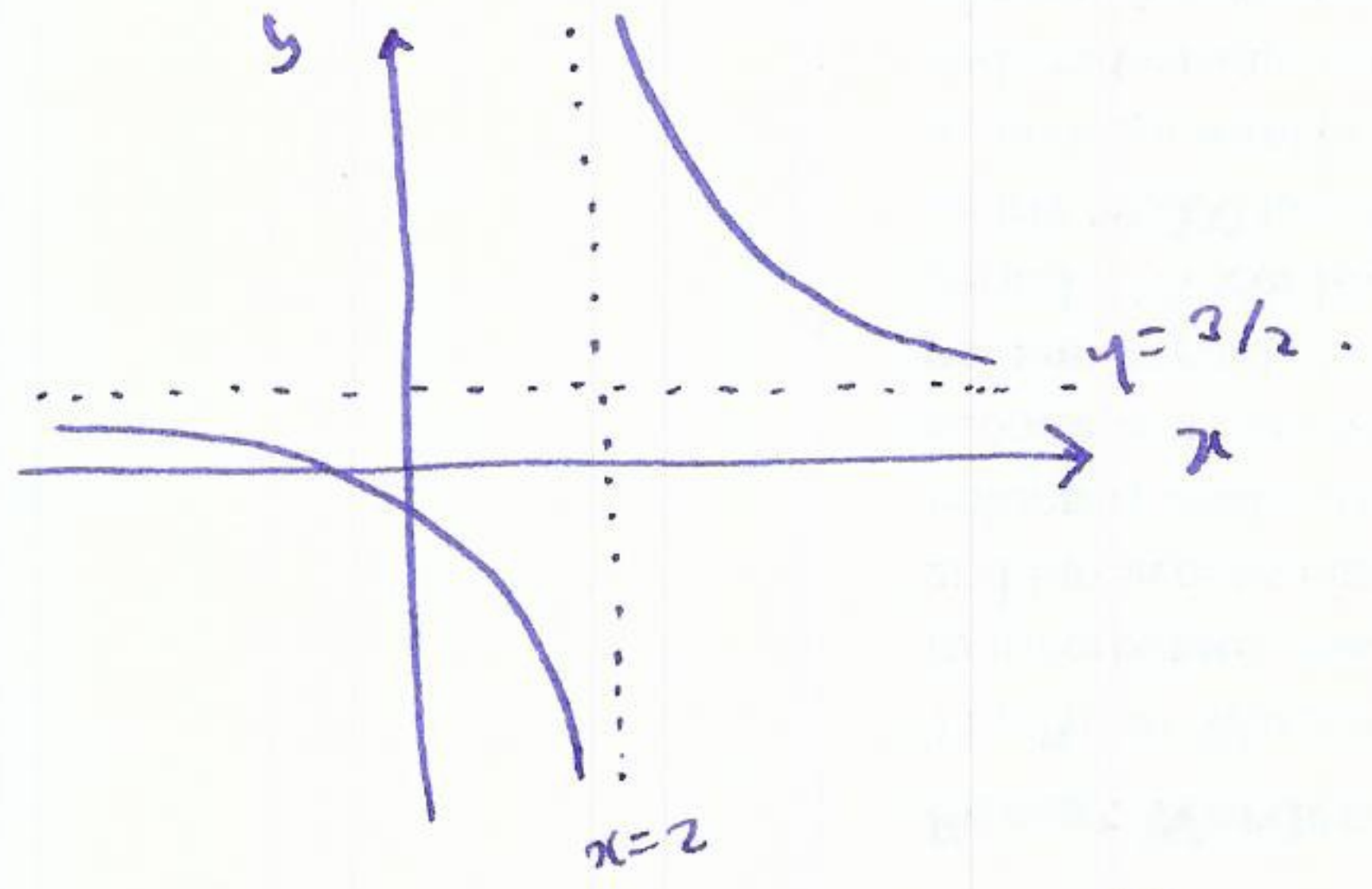
$2x-4=0 \Rightarrow x=2$

② find $f'(x) = \frac{(2x-4) \cdot 3 - (3x+2) \cdot 2}{(2x-4)^2} = \frac{-16}{(2x-4)^2} = \frac{-4}{(x-2)^2} \leftarrow \text{always -ve decreasing}$

no critical points

③ find $f''(x) = \frac{8}{(x-2)^3}$ +ve $x > 2$ concave up
-ve $x < 2$ concave down.

④ horizontal asymptote: $f(x) = \frac{3+4/x}{2-4/x} \quad \lim_{x \rightarrow \pm\infty} f(x) = \frac{3}{2}$



behavior near asymptote.

$x = 2 + \epsilon \quad \frac{3x+2}{2(x-2)} + > 0 \rightarrow +\infty$

$x = 2 - \epsilon \quad \frac{3x+2}{2(x-2)} - < 0 \rightarrow -\infty$

Example sketch graph of $\frac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-1/2}$.

① vertical asymptotes: none.

② find $f'(x) = x \cdot \frac{-1}{2}(x^2+1)^{-3/2} + (x^2+1)^{-1/2} \cdot 1 = \frac{\sqrt{x^2+1} - x^{\frac{1}{2}}(x^2+1)^{-1/2} \cdot 2x}{x^2+1} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0$ increasing

③ find $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x$ point of inflection at 0
+ve for $x < 0$ concave up
-ve for $x > 0$ concave down.

④ horizontal asymptotes: $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1+1/x^2}} = 1$