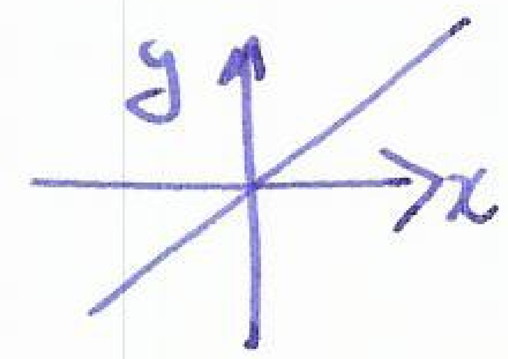
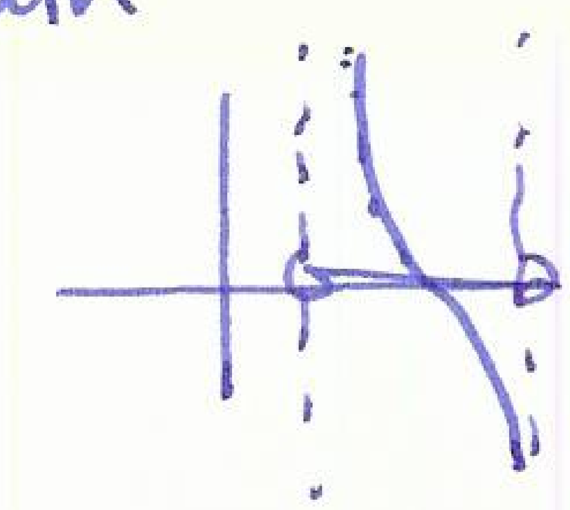
warning: some functions don't have any max/min

example $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$



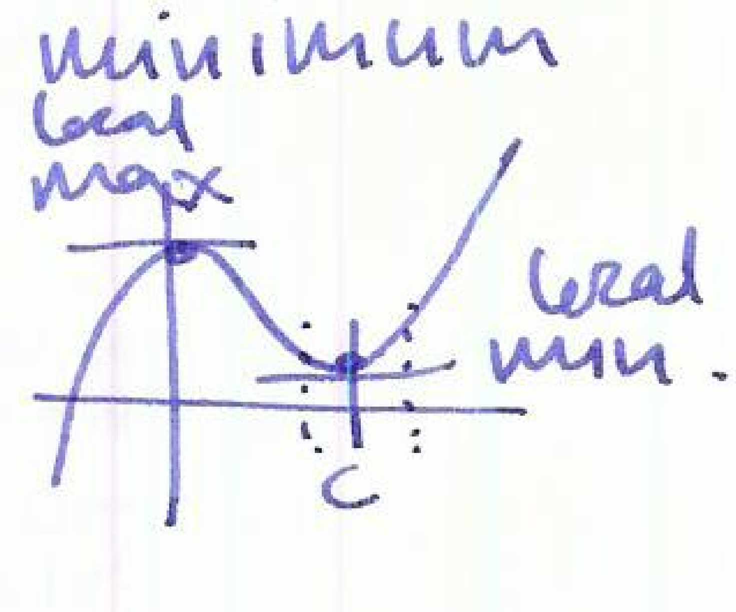
open intervals (a,b)



Thm If $f(x)$ is continuous on a closed bounded interval, then $f(x)$ has both an absolute max and absolute min.

Defn $f(x)$ has a local min at $x=c$ if $f(c)$ is the minimum value of $f(x)$ for some small interval containing c.

similarly for local max.

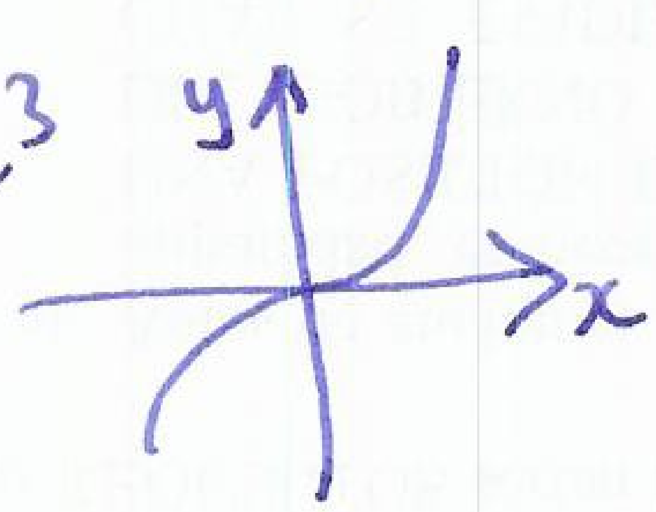


Defn c is a critical point if $f'(c) = 0$

Thm If c is a local max or min, then $f'(c) = 0$.

warning $f'(c) = 0 \not\Rightarrow$ local max or min

Example $y = x^3$



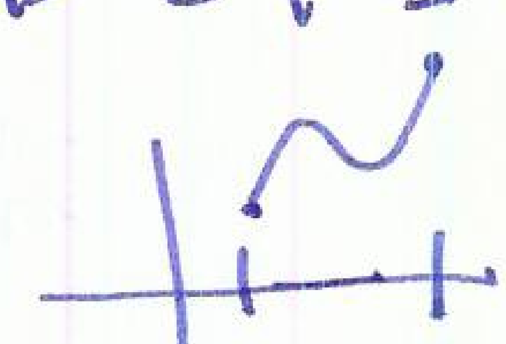
$f'(x) = 2x^2 = 0$ when $x = 0$.

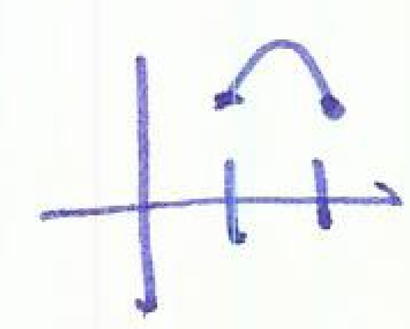
but $x=0$ not local max or min.

How to find absolute max/min of functions on closed intervals $[a,b]$.

① find critical points, and evaluate function there

② check endpoints.





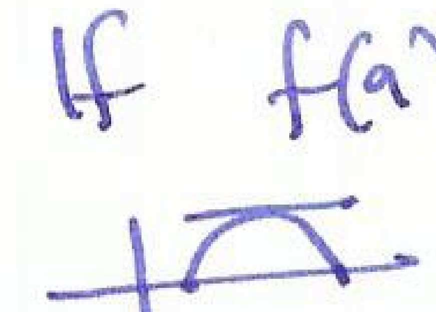
Example find absolute max/min of $2x^3 - 15x^2 + 24x + 7$ on $[0,6]$

$x^2 - 8$ on $[1,4]$

$\sin x \cos x$ on $[0,\pi]$.

Thm (Rolle's thm) Suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$ then there is a $c \in (a,b]$ s.t. $f'(c) = 0$

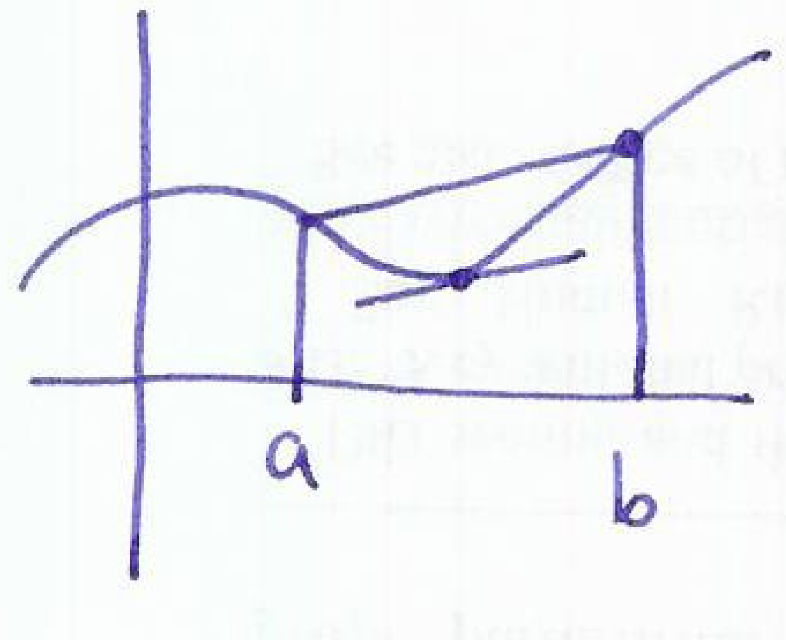
Proof



if there is a local max/min then $\exists c$ s.t. $f'(c) = 0$

if no local max/min $\Rightarrow f(x) = \text{const} \Rightarrow f'(c) = 0 \; \forall c$.

§4.3 Mean value theorem and monotonicity / first derivative test



Thm Mean Value Theorem (MVT)

Suppose f is cts on $[a,b]$ and differentiable on (a,b), then there is a $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

i.e. there is a point where the slope is equal to the average rate of change.

Corollary If $f(x)$ is differentiable, and $f'(x)=0$, then $f(x)=c$ (constant)

Proof suppose there is a,b s.t. $f(a) \neq f(b)$. Then $\exists\, c \in (a,b)$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a} \neq 0$. □

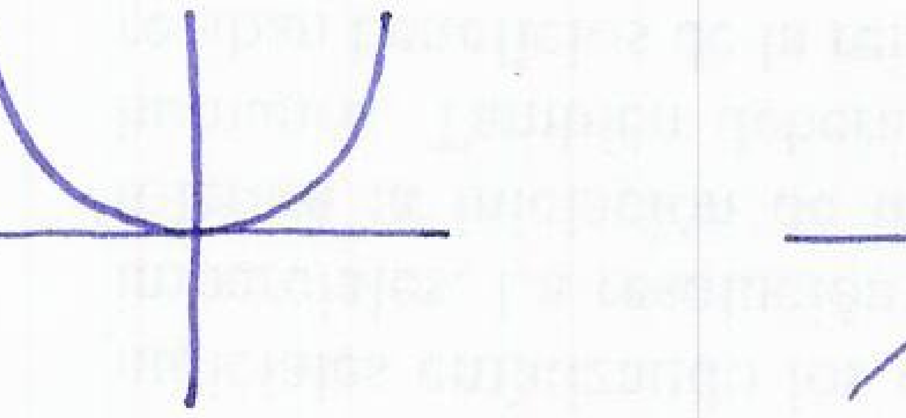
Monotonicity

Suppose f is differentiable on (a,b):

If $f'(x) > 0$ for all $x \in (a,b)$ then f is increasing on (a,b)

$f'(x) < 0$ decreasing on (a,b)

Example ① $f(x) = x^2$ $\quad f'(x) = 2x$



increasing on $(0,\infty)$

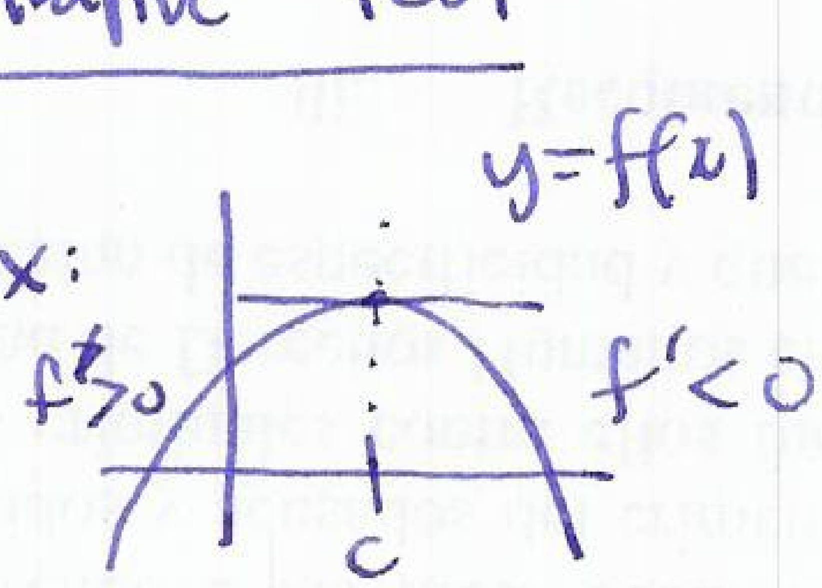
decreasing on $(-\infty,0)$

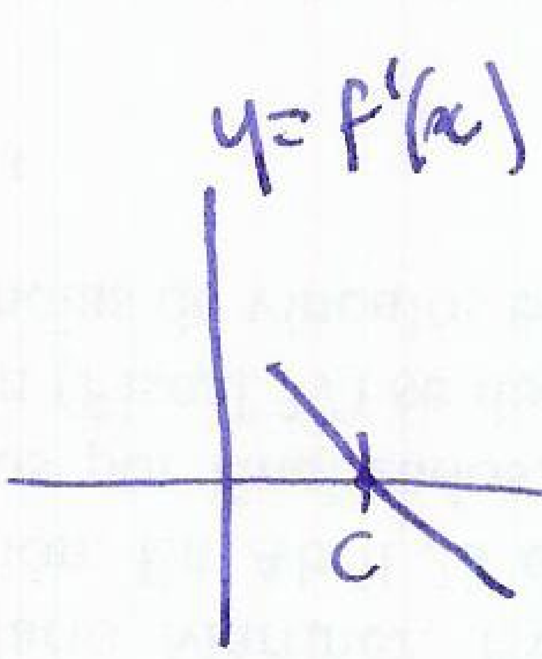
② where is $f(x) = x^2-2x-3$ increasing? $f'(x) = 2x-2$

$f'(x) > 0$ when $2x-2>0$ $\quad x>1$.

First derivative test

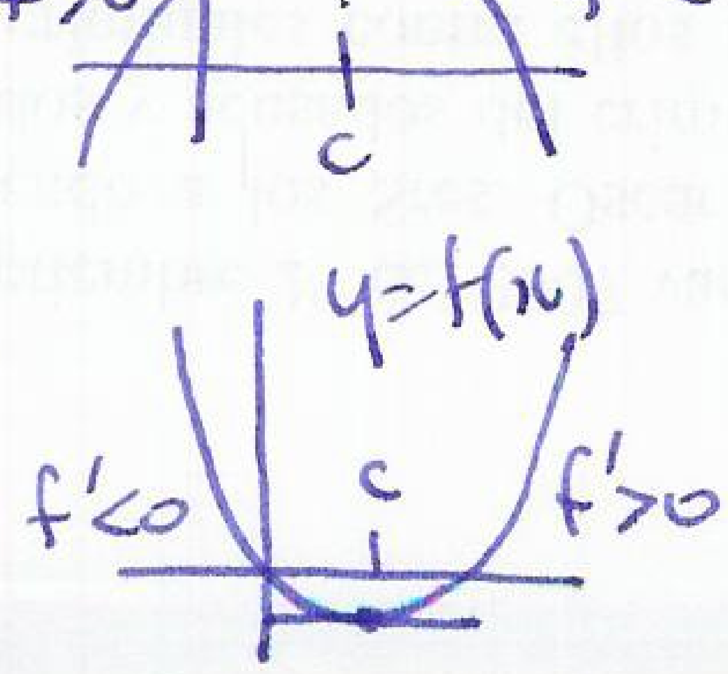
local max:

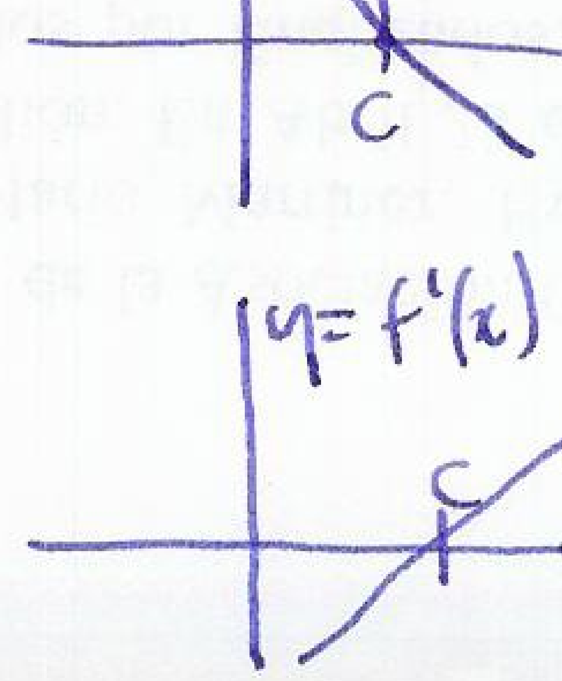




$f'(x)$ goes from +ve to −ve $\Rightarrow$ local max

local min:





$f'(x)$ goes from −ve to +ve $\Rightarrow$ local min.

Thm First derivative test

If $f(x)$ differentiable and $f'(c)=0$ (i.e. c is a critical point)

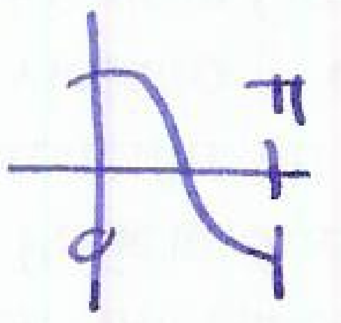
then if $f'(x)$ changes from +ve to −ve at c $\Rightarrow$ local max

−ve   +ve $\Rightarrow$ local min.

Example classify critical points of $f(x) = \cos^2 x + \sin x$ on $[0,\pi]$

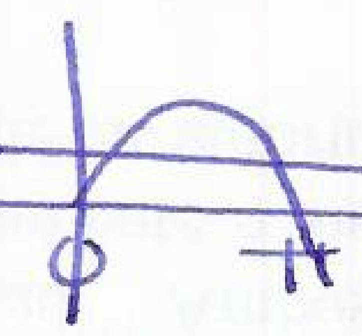
find critical points : $f'(x) = -2\cos x \cdot \sin x + \cos x$

solve $f'(x)=0$: $\cos x\,(1-2\sin x) = 0$ $\cos x = 0$



$\Rightarrow x = \frac{\pi}{2}$

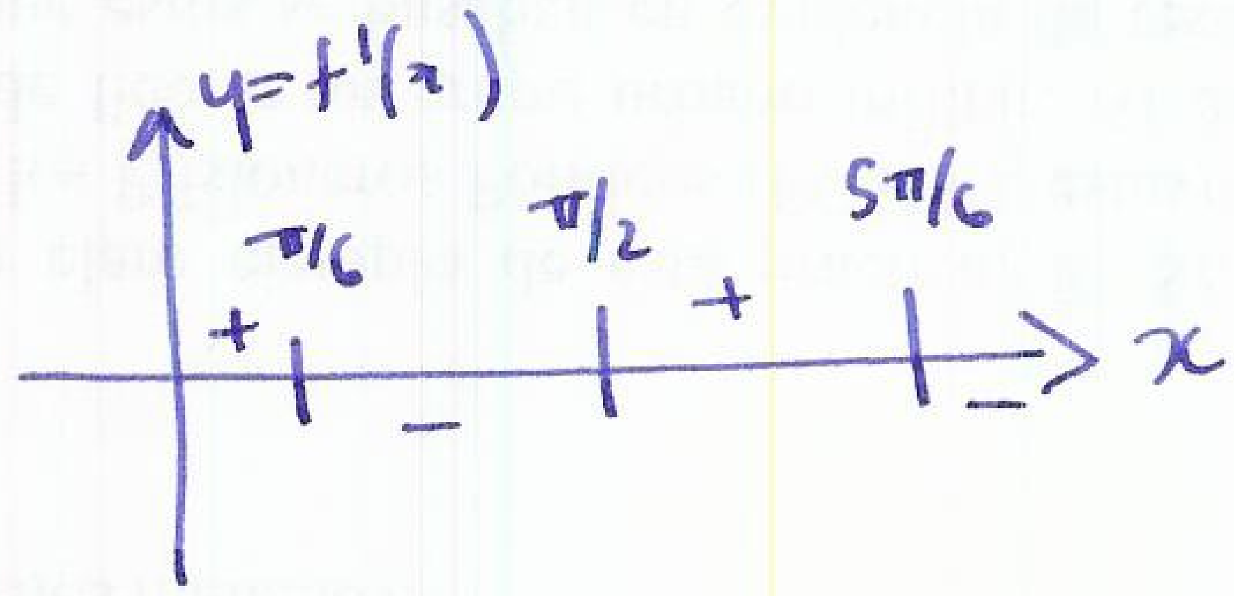
$1-2\sin x = 0 \Leftrightarrow \sin x = \frac{1}{2}$



$x = \frac{\pi}{6}, \frac{5\pi}{6}$.

critical points: $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

find sign of $f'(x)$ in between

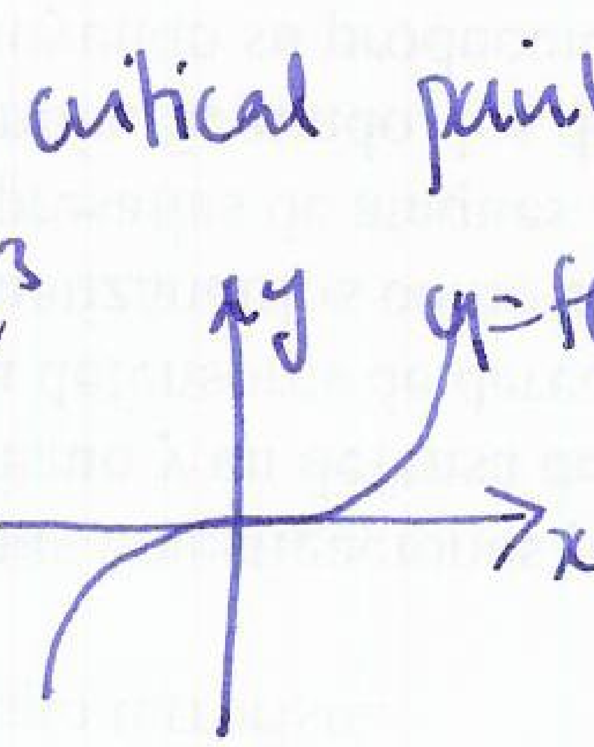


$\Rightarrow$ local max : $\pi/6, 5\pi/6$

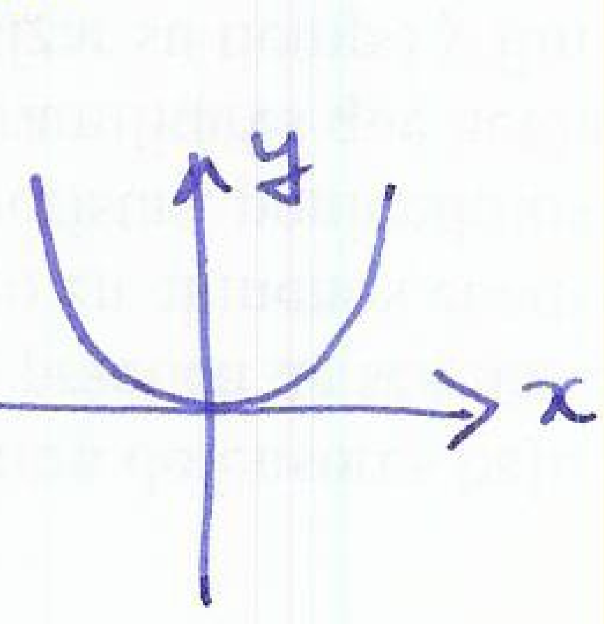
local min : $\pi/2$

Example critical point not max or min

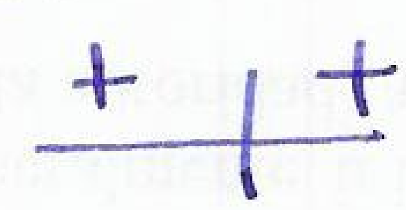
$f(x) = x^3$



$f'(x) = 3x^2$



$f'(0) = 0$



$\not\Rightarrow$ not max or min.