③ $f(x) = \ln(x^2) = 2\ln(x)$ $\qquad$ $f(x) = (\ln(x))^2$

$f'(x) = \frac{2}{x}$ $\qquad$ $f'(x) = 2\ln(x) \cdot \frac{1}{x}.$

~~Example logarithmic~~

Hyperbolic trig functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}(\sinh(x)) = \cosh(x) \qquad \frac{d}{dx}(\cosh(x)) = \sinh(x) \qquad \boxed{\cosh^2 x - \sinh^2 x = 1}$$

check: $(\sinh(x))' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh(x).$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ etc....

$$\frac{d}{dx}(\tanh(x)) = \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)}$$

$$= \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x).$$

Inverse functions

$y = \sinh^{-1}(x)$

$\sinh(y) = x$

$\cosh(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{1+x^2}}$

summary

$$\frac{d}{dx}(\sinh^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}$$

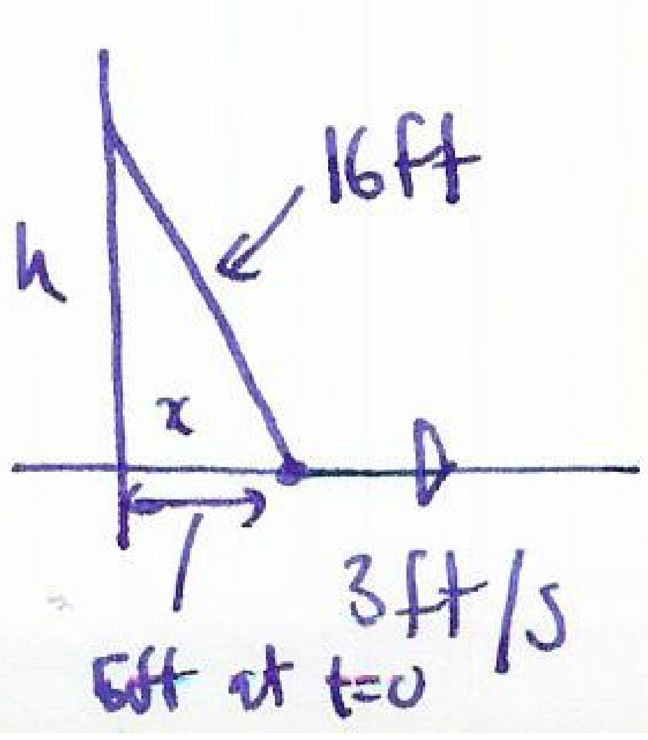
$$\frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1}(x)) = \frac{1}{1-x^2}$$

§3.11 Related rates

Example: falling ladder $\qquad$ $x(t)$ distance of foot of ladder from wall

$h(t)$ height of top of ladder.



Q: if $\frac{dx}{dt} = 3$ft/s what is $\frac{dh}{dt}$? (when you hit the ground?)

always true (for all t): $\qquad$ true at $t = 0$:

$x^2 + h^2 = 16^2$ $\qquad$ $x(0) = 5$

$\frac{dx}{dt} = 3.$

$x^2 + h^2 = 16^2$

$(x(t))^2 + (h(t))^2 = 16^2$

diff. wrt t: $\quad 2x(t)\underbrace{\frac{dx}{dt}}_{3.} + 2h(t)\frac{dh}{dt} = 0$

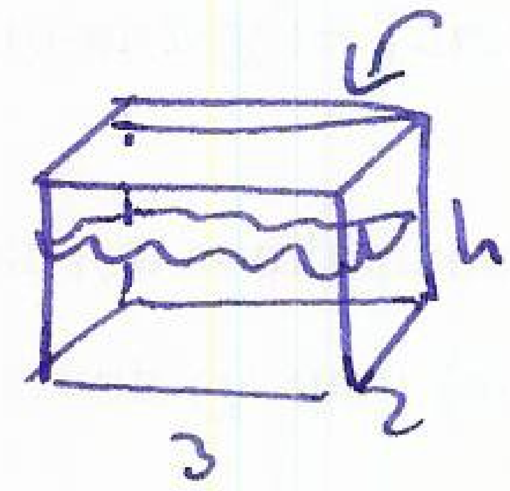
$\left.\begin{aligned} x(t) &= 5+3t \\ h(t) &= \sqrt{16^2 - (5+3t)^2} \end{aligned}\right\}$ so $\dfrac{dh}{dt} = -\dfrac{x(t)}{h(t)} = -\dfrac{5+3t}{\sqrt{16^2-(5+3t)^2}}$.

hit ground at $5+3t = 16 \Rightarrow t = 11/3$

so $\dfrac{dh}{dt}(11/3) = -\dfrac{16}{\sqrt{16^2-16^2}}$ i.e. $\dfrac{dh}{dt}(t) \to -\infty$ as $t \to \dfrac{11}{3}$.

Example: filling a rectangular tank

3 ft³/min



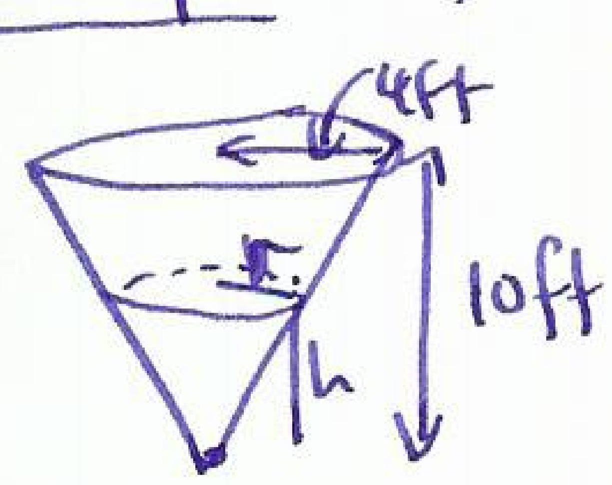
how fast is the water rising?

$V(t)$ = volume of water

$h(t)$ = height.

$V = 6h \qquad \dfrac{dV}{dt} = 6\dfrac{dh}{dt}$ so $\dfrac{dV}{dt} = 3 = 6\dfrac{dh}{dt} = \dfrac{1}{2}$ ft/min

Example: filling a conical tank



water in at 10 ft³/min $\leftrightarrow \dfrac{dV}{dt} = 10$ ft³/min

volume of cone $V = \frac{1}{3}\pi h r^2$

need r in terms of h: $\dfrac{r}{h} = \dfrac{4}{10} = \dfrac{2}{5} \Rightarrow r = \dfrac{2}{5}h$.

so $V = \dfrac{1}{3}\pi h\left(\dfrac{2}{5}h\right)^2 = \dfrac{4}{75}\pi h^3$

$\dfrac{dV}{dt} = \dfrac{12}{75}\pi h^2 \dfrac{dh}{dt}$ so when $h = 5$, water rising at rate: $\dfrac{dh}{dt}$

$10 = \dfrac{12}{75}\pi . (5)^2 \dfrac{dh}{dt} \qquad \dfrac{dh}{dt} = \dfrac{10}{4\pi}$ ft/min