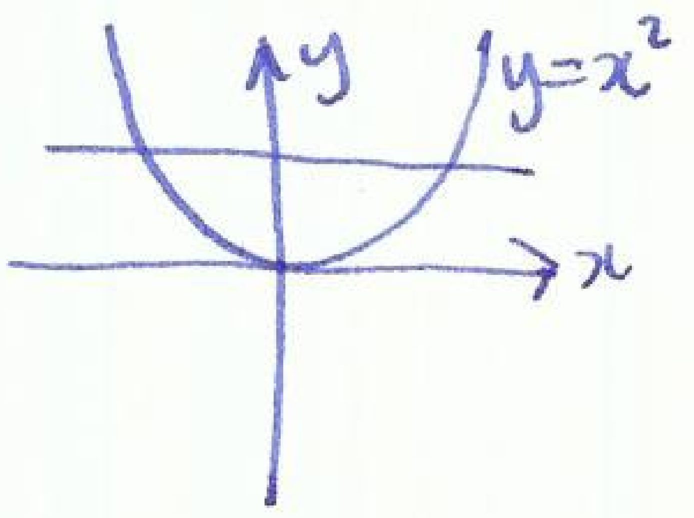
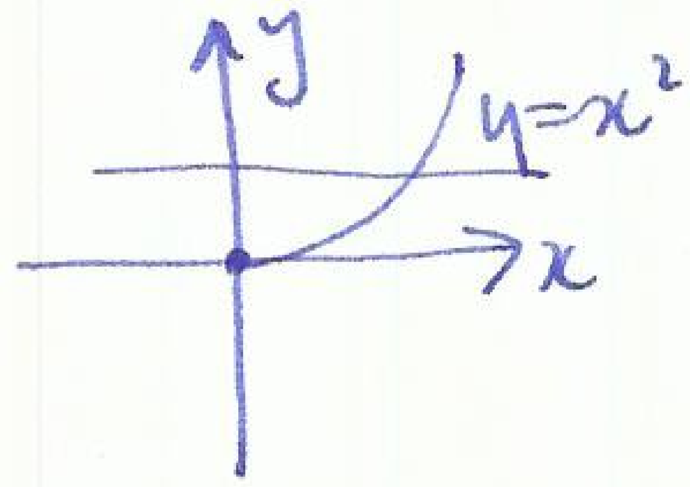
bad example $f(x)=x^2$ problems: doesn't pass horizontal line test.



fix: restrict domain consider $f:[0,\infty) \to [0,\infty)$

$x \longmapsto x^2$

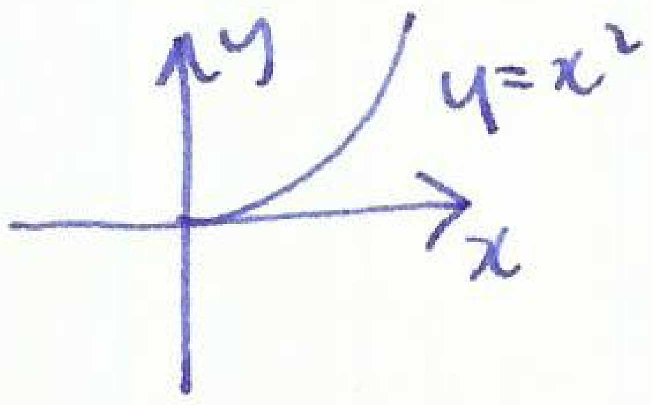


this does pass the horizontal line test, so there is an inverse called $\sqrt{x}$.

$f^{-1}: [0,\infty) \to [0,\infty)$

$x \longmapsto \sqrt{x}$.

How to draw the graph of the inverse



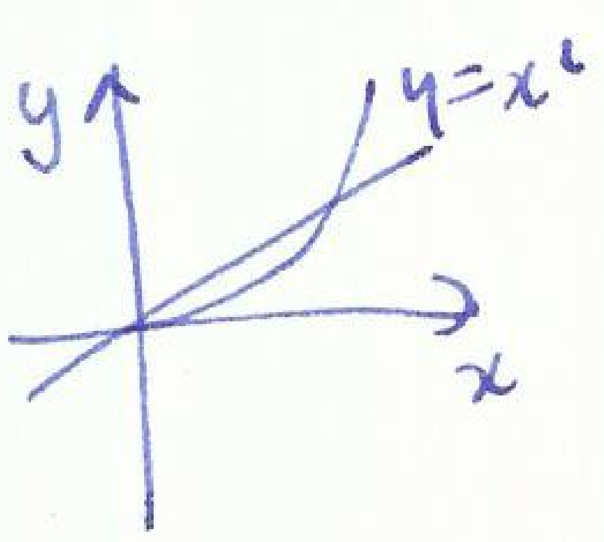
graph of $f(x)$: set of pairs $\overset{(x,y)}{(x, f(x))}$ $x \in$ domain of f

graph of $f^{-1}(x)$: set of pairs $(x, f^{-1}(x))$ $x \in$ domain ~~range~~ of f^{-1} $=$ range of f.

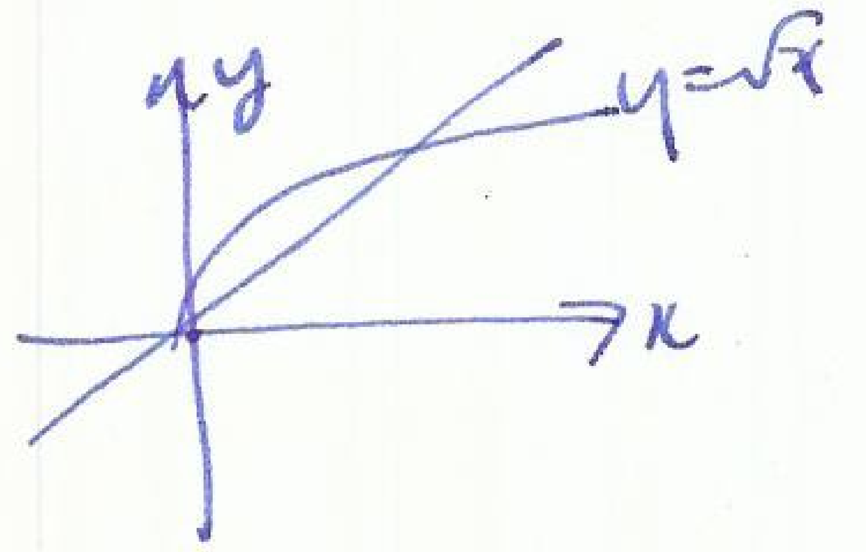
normally call points in range of f y so want $(y, f^{-1}(y))$

but if $y=f(x) \Leftrightarrow f^{-1}(y)=x$ so want $(y, x) \Leftrightarrow (f(x), x) \Leftrightarrow (y, f^{-1}(y))$

summary $\boxed{\text{graph of } f^{-1}(x) \text{ is graph of } f(x) \text{ reflected in } y=x}$.

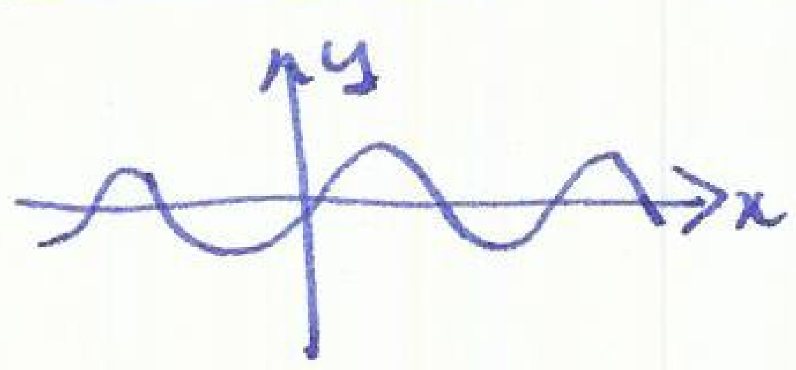


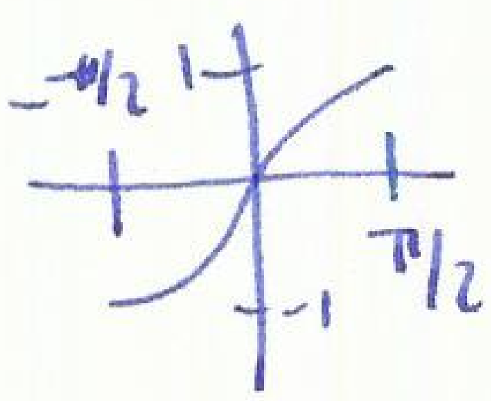
reflect



Inverse trig functions

$y=\sin(x)$





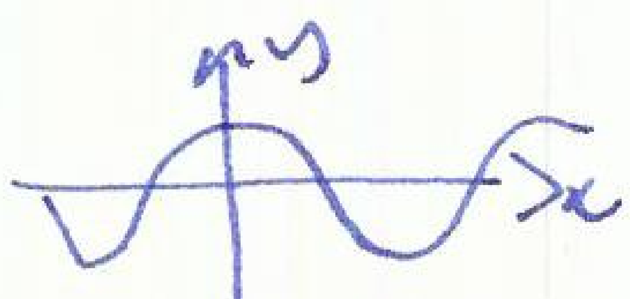
problem: not one-to-one

fix: restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(x): [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

$\sin^{-1}(x) : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$
$\|$
$\arcsin(x)$

similarly $y=\cos(x)$



restrict domain to $[0,\pi]$.

$\cos(x) : [0,\pi] \to [-1,1]$

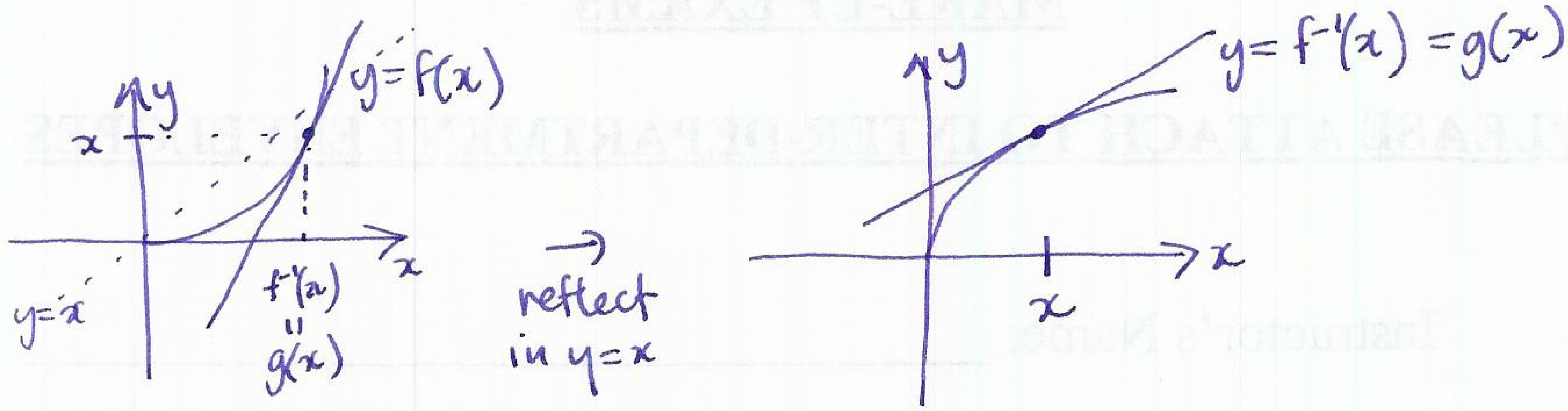
$\cos^{-1}(x) : [-1,1] \to [0,\pi]$.

§4.9 Derivatives of inverse functions

Thm $f(x)$ differentiable and one-to-one, with inverse $f^{-1}(x) = g(x)$.

Then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

recall



$\dfrac{1}{\text{slope}}$ at $f^{-1}(x) = g(x)$ ⟵ slope at x

$\dfrac{1}{f'(g(x))}$. $g'(x)$

Example $f(x) = x^2$

$g(x) = f^{-1}(x) = \sqrt{x}$

$f'(x) = 2x \qquad g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}x^{-1/2}$

Alternate method (implicit differentiation)

$y = f^{-1}(x)$

$f(y) = x$

diff implicitly wrt x: $\quad f'(y)\dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Thm $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad$ $\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = -\dfrac{1}{\sqrt{1-x^2}}$

Proof (of $\sin^{-1}(x)$)

$$y = \sin^{-1}(x)$$

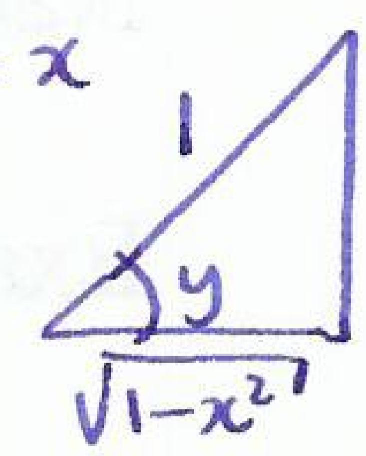
$$\sin(y) = x$$

$$\cos(y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \qquad \text{use: } \sin^2(y) + \cos^2(y) = 1$$

$$\cos(y) = \sqrt{1-\sin(y)} = \sqrt{1-x^2}.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \text{or triangle trick: } \sin(y) = x$$



$\square$

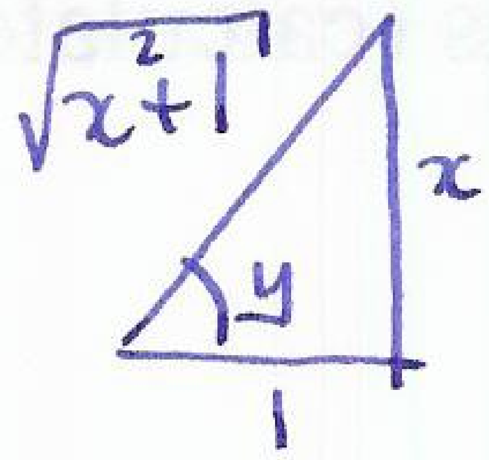
Thm $\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{x^2+1}$ $\qquad$ $\dfrac{d}{dx}\left(\cot^{-1}(x)\right) = \dfrac{-1}{x^2+1}$

$\dfrac{d}{dx}\left(\sec^{-1}(x)\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$ $\qquad$ $\dfrac{d}{dx}\left(\operatorname{cosec}^{-1}(x)\right) = \dfrac{-1}{|x|\sqrt{x^2-1}}$.

Proof (of $\tan^{-1}(x)$)

$$y = \tan^{-1}(x)$$

$$\tan(y) = x$$



$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{x^2+1} \quad \square$$

Example $\dfrac{d}{dx}\left(\tan^{-1}(e^{2x})\right) = \dfrac{1}{1+(e^{2x})^2} \cdot 2e^{2x}$

§3.10 Derivatives of exponentials and logs

recall: $f(x) = b^x$ then $f'(x) = m_b b^x$

$f(x) = e^x$ then $f'(x) = e^x$

Thm $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$

Proof $f(x) = b^x = e^{x\ln(b)}$

$f'(x) = e^{x\ln(b)} \cdot \ln(b)$ (chain rule)

$= b^x \ln(b)$. $\square$

Example $f(x) = 3^{4x}$ $\quad f'(x) = 4\ln(3)\cancel{4}3^{4x}$

Thm $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$

Proof $y = \ln(x)$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ $\square$

Example ① $f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}$

$f'(x) = \frac{1}{x\ln(b)}$

② $f(x) = x\ln(x)$

$f'(x) = \ln(x) + x\frac{1}{x} = \ln(x) + 1$