Proof (of chain rule)

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$ [answer should be $f'(g(x))g'(x)$]

so write this as: $$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \underbrace{\frac{g(x+h)-g(x)}{h}}_{\text{this gives } g'(x)} \quad (*)$$

set $k = g(x+h) - g(x)$

then as g is continuous $h \to 0 \Rightarrow k \to 0$

so $$\lim_{k \to 0} \frac{f(g(x)+k) - f(g(x))}{k} = f'(g(x)).$$

so $(*) = f'(g(x)).g'(x)$ as required $\square$

More examples $$\frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1}.g'(x)$$

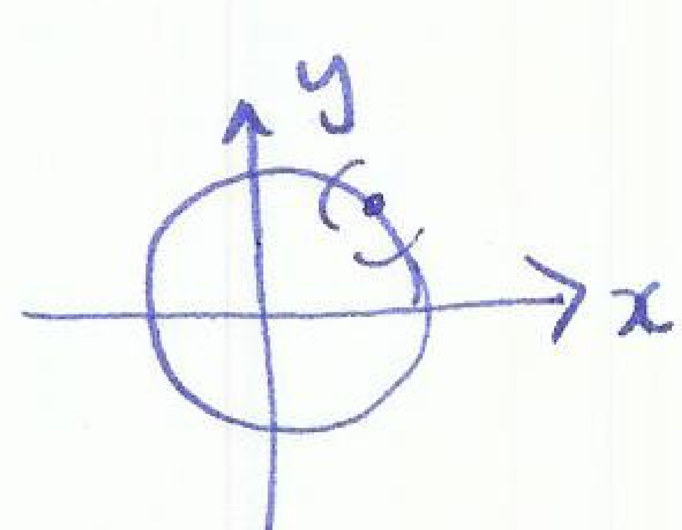
$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}.g'(x)$$

$$\frac{d}{dx}\left(f(ax+b)\right) = af'(ax+b)$$

§3.8 Implicit differentiation

explicit function: $f(x) = x^2+x+1$

implicit "function": $x^2+y^2=1$

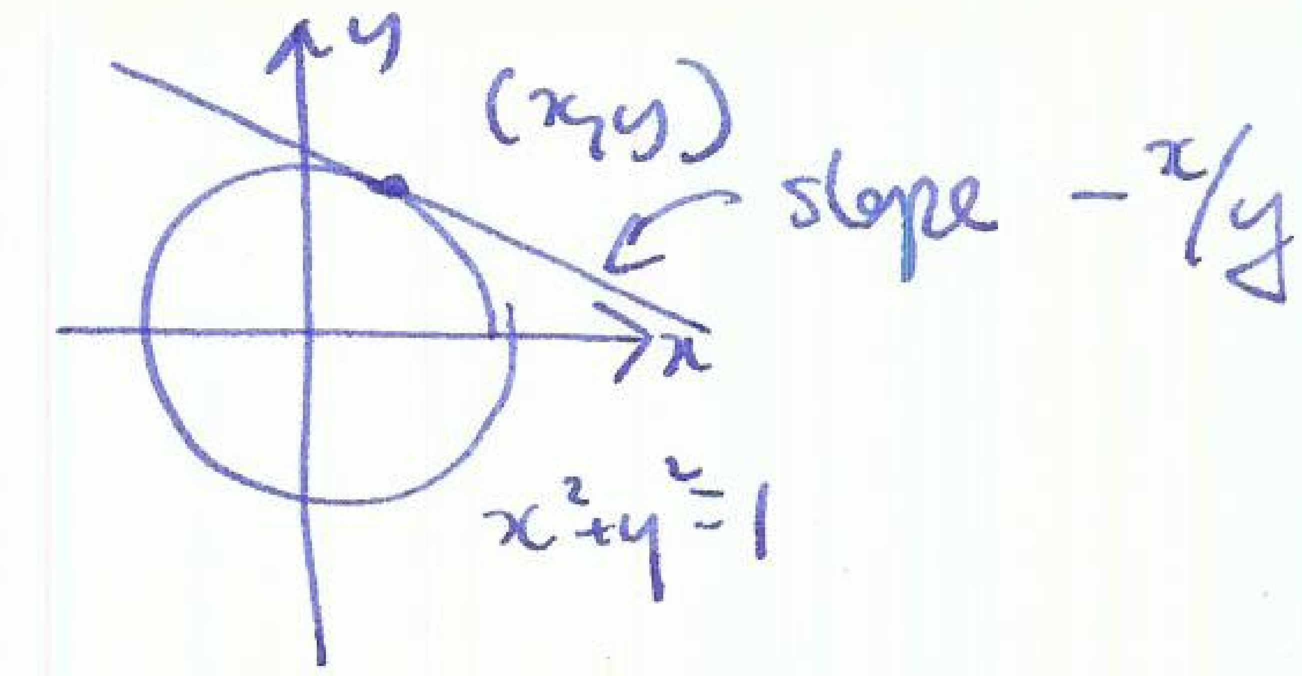


"local" function near a point (x,y) on the curve we can think of y as a function of x, i.e. $(x, y(x))$

differentiate $x^2+y^2=1$ wrt to x

$$x^2 + (y(x))^2 = 1$$

$$\frac{d}{dx}\left(x^2+(y(x))^2 \right) = \frac{d}{dx}(1)$$



$$2x + 2y(x).y'(x) = 0$$

$$y'(x) = -\cancel{\tfrac{2x}{2y}}\; x/y$$

Example ① find tangent line to circle $x^2+y^2=1$ at $\left(\frac{3}{5},\frac{4}{5}\right)$

$$\frac{dy}{dx} = -x/y = \frac{-3/5}{4/5} = \frac{-3}{4} \qquad y-\frac{4}{5} = -\frac{3}{4}\left(x-\frac{3}{5}\right)$$

② find tangent line to $y^4+xy = x^3-x+2$ at $(1,1)$

implicit differentiation: $4yy' + y + xy' = 3x^2-1$

at $x=1, y=1$: $4y'+1+y' = 3-1$

$$5y' = 2 \quad y' = \frac{2}{5} \qquad y-1 = \frac{2}{5}(x-1)$$

§1.5 Inverse functions

recall $f: X \longrightarrow Y$
domain range
$x \longmapsto f(x)$

want: the inverse function should be the reverse of this.

$$X \xleftarrow{f^{-1}} Y$$

problem: the inverse is often not a function

$a, b \to f(a)=f(b)$

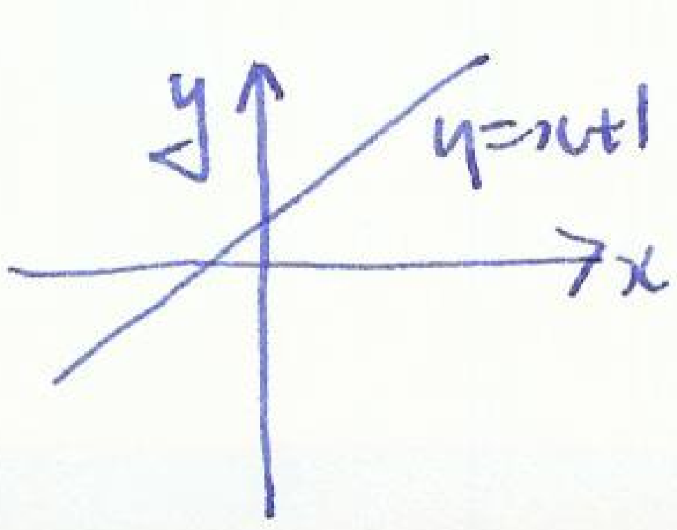
suppose there is $a \neq b$ such that $f(a)=f(b)$
what is $f^{-1}(f(a))$?

Q: when does a function have an inverse?

A: when its one-to-one (injective) $\Leftrightarrow$ passes the horizontal line test

$\Leftrightarrow$ for each value ~~f(x)~~ c $\in$ range, there is a unique $x \in$ domain s.t. $f(x) = c$.

Example $y=x+1$



Q: how do we find a formula for the inverse?

A: ① write down $y=f(x)$
② solve for x in terms of y, i.e. $x=g(y)$
③ $f^{-1}(x) = g(x)$.
④ check!