note: slope at $x=0$ is $f'(0) = m_b b^0 = m_b$

recall: e is defined to be the number such that the slope of e^x at $x=0$ ~~ff~~ is 1. ~~That~~ Therefore if $\boxed{f(x) = e^x \quad \text{then } f'(x) = e^x}$

$\Leftrightarrow \boxed{\frac{d}{dx}(e^x) = e^x}$

Example differentiate $7e^x + 4x^2$

answer $7e^x + 8x$

Thm Differentiable $\Rightarrow$ continuous.

Proof $f(x)$ differentiable at x means $\lim\limits_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists.

want to show $\lim\limits_{h \to 0} f(c+h) = f(c)$

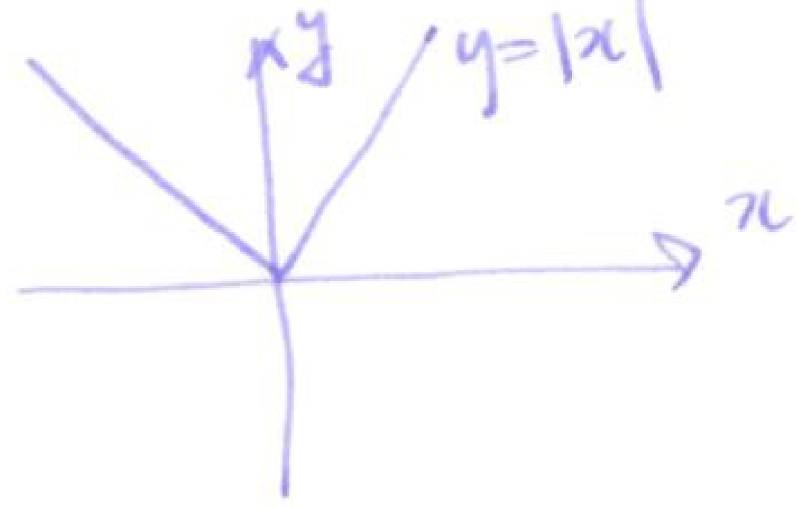
consider $f(c+h) - f(c) = h\left(\frac{f(c+h) - f(c)}{h}\right)$

but $\lim\limits_{h \to 0} h = 0$ and $\lim\limits_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$ same number

of $\lim\limits_{h \to 0} h\left(\frac{f(c+h) - f(c)}{h}\right) = 0 \cdot f'(c) = 0$ (product rule) $\square$.

Example Continuous but not differentiable (everywhere)

$f(x) = |x|$



continuous (we did this earlier?).

differentiable? $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{|x+h| - |x|}{h}$

at $x=0$: $\lim\limits_{h \to 0} \frac{|h|}{h}$ DNE.

local picture: if $f(x)$ differentiable at $x=c$, then if you close enough graph looks like straight line.

§3.3 Product and quotient rules

new functions from old: $f(x)\,g(x)$ product $\qquad f(x)/g(x)$ quotient

Thm Product rule

$$(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}\,g + f\,\frac{dg}{dx}$$

Warning: $(fg)' \neq f'g'$!!

Example ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x).x + x.\frac{d}{dx}(x) = 1.x + x.1 = 2x$

② $$\frac{d}{dx}\left(3x^2(x^2+1)\right) = \frac{d}{dx}(3x^2).(x^2+1) + 3x^2\frac{d}{dx}(x^2+1).$$
$$= 6x(x^2+1) + 3x^2.2x$$

③ $$\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2).e^x + x^2.\frac{d}{dx}(e^x)$$
$$= 2xe^x + x^2e^x$$

Proof (of product rule) (assume f, g both differentiable at x)

$$(fg)'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

trick: $$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h).\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x).\frac{f(x+h)-f(x)}{h} \quad \text{(sum)}$$

$$= \underbrace{\lim_{h\to 0} f(x+h)}_{\substack{f(x) \\ (\text{differentiable} \Rightarrow \text{cts})}} \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{\substack{g'(x) \\ (\text{defn})}} + \underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{f'(x)\ (\text{defn})} \underbrace{\lim_{h\to 0} g(x)}_{g(x)} \quad (\text{product})$$

$$= f(x).g'(x) + f'(x).g(x) \quad \square$$

<u>Thm</u> <u>Quotient rule</u> (assume f, g differentiable, $g(x) \neq 0$)

Then $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x).f'(x) - g'(x)f(x)}{(g(x))^2}$

<u>Example</u> ① $\dfrac{d}{dx}\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1).(x)' - (x+1)'.x}{(x+1)^2}$

$$= \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

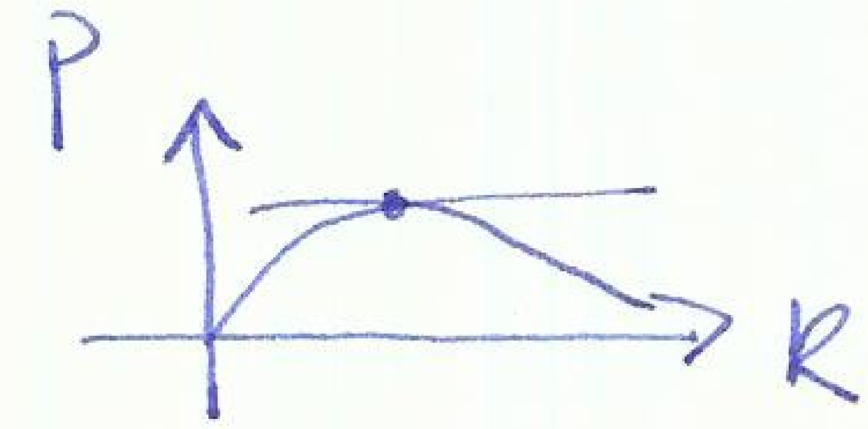
② $\dfrac{d}{dt}\left(\dfrac{e^t}{e^t+1}\right) = \dfrac{(e^t+1)(e^t)' - (e^t+1)'e^t}{(e^t+1)^2} = \dfrac{(e^t+1)e^t - e^t.e^t}{(e^t+1)^2}.$

$$= \frac{e^t}{(e^t+1)^2}$$

③ battery power

r internal resistance

R resistance power $P = \dfrac{V^2 R}{(r+R)^2}$



<u>Q</u>: when does the battery give maximum power?

<u>A</u>: when $\dfrac{dP}{dr} = 0$

$$P = \frac{V^2 R}{(r+R)^2} \qquad P(R) \quad V, r \text{ constant}$$

$$\frac{dP}{dr} = \frac{(r+R)^2 \cdot (V^2R)' - ((r+R)^2)' . V^2 R}{(r+R)^4} = \frac{(r+R)^2 V^2 - (r^2+2rR+R^2)' V^2 R}{(r+R)^4}$$

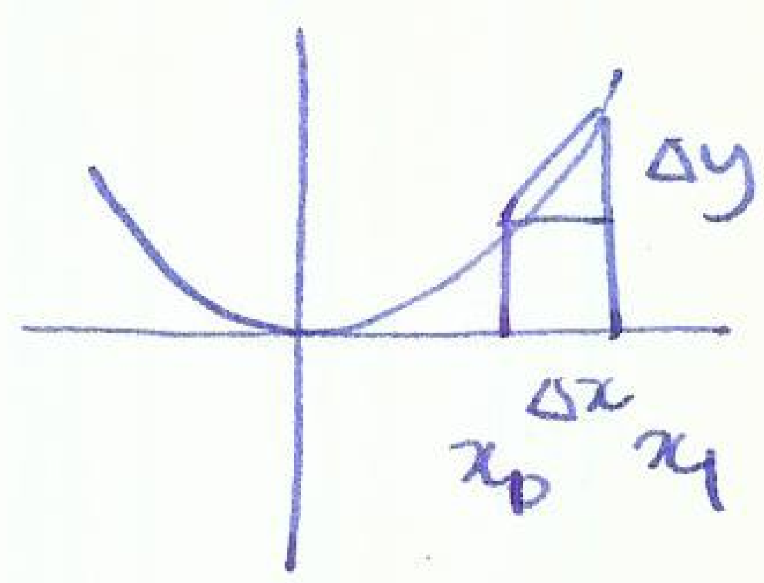
$$= \frac{V^2\left[(r+R)^2 - (2r+2R)R\right]}{(r+R)^4} = \frac{V^2\left[r^2+2rR+R^2 - 2rR - 2R^2\right]}{(r+R)^4}$$

$$= \frac{V^2(r^2-R^2)}{(r+R)^4} = V^2\frac{r-R}{(r+R)^4} = 0 \text{ when } r=R. \quad \square$$

§3.4 Rates of change

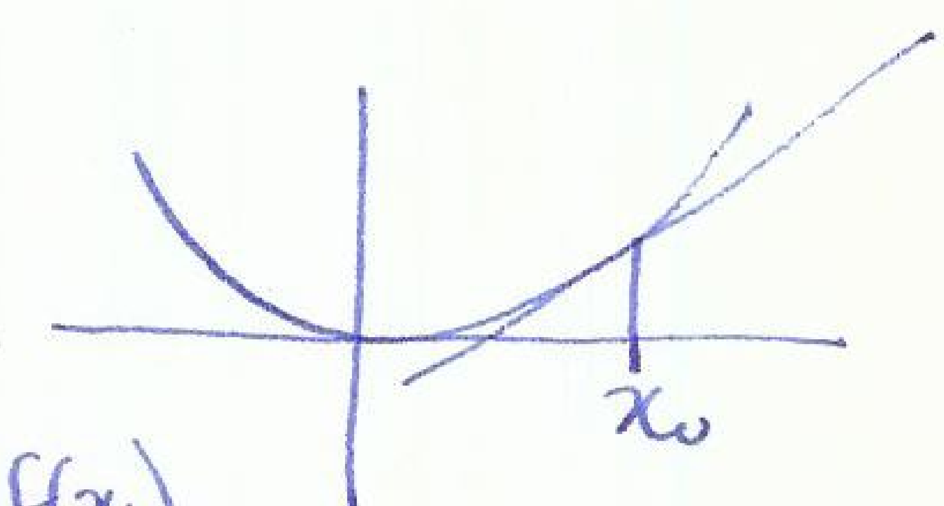
recall: average rate of change

$$= \frac{\Delta x}{\Delta y} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$



(instantaneous) rate of change

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y} = \lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

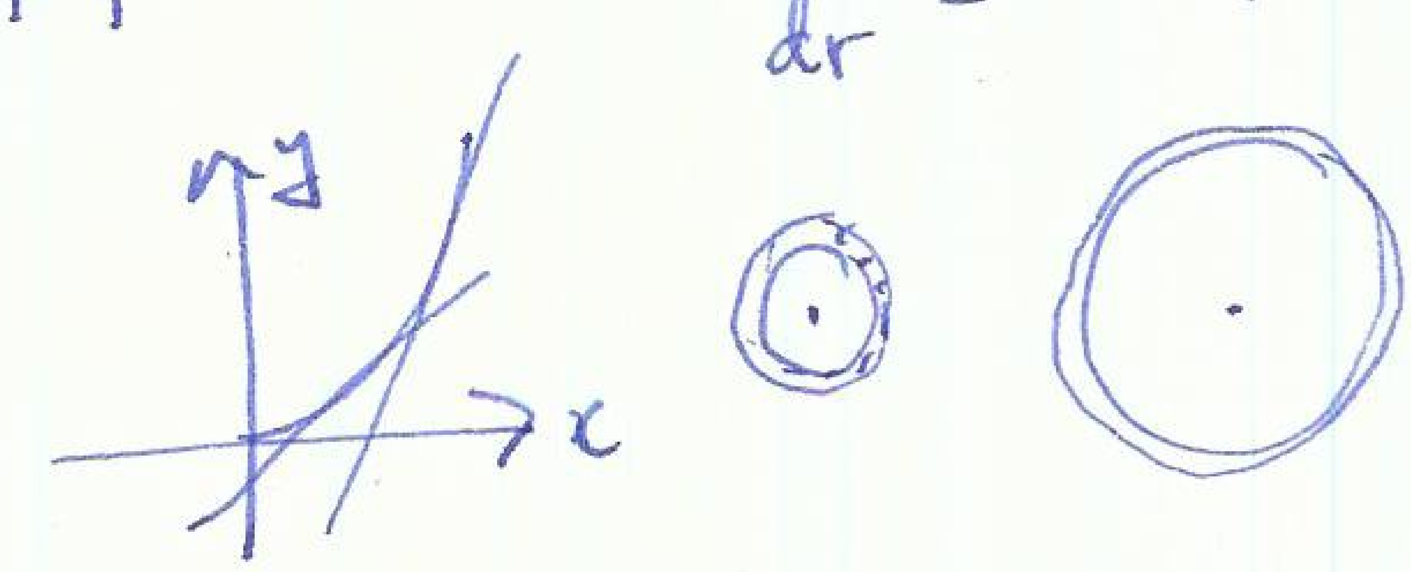


observation: if Δx is small, we can use average RoC to approximate actual RoC and vice versa.

Example: area of circle $A = \pi r^2$

calculate rate of change of area with respect to radius: $\frac{dA}{dr} = 2\pi r$

e.g. $\left.\frac{dA}{dr}\right|_{r=2} = 4\pi \qquad \left.\frac{dA}{dr}\right|_{r=5} = 10\pi$



For small h, $f'(x_0) \approx \dfrac{f(x_0+h) - f(x_0)}{h}$ or $f(x_0+h) \approx f(x_0) + hf'(x_0)$

<u>Example</u> stopping distance in feet given by $F(s) = 1.1s + 0.05s^2$
(s speed in mph)

calculate stopping distance when $s = 30$: $F(30) = 1.1 . 30 + 0.05(30)^2$
$= 78$

calculate RoC of stopping distance wrt speed when $s=30$:

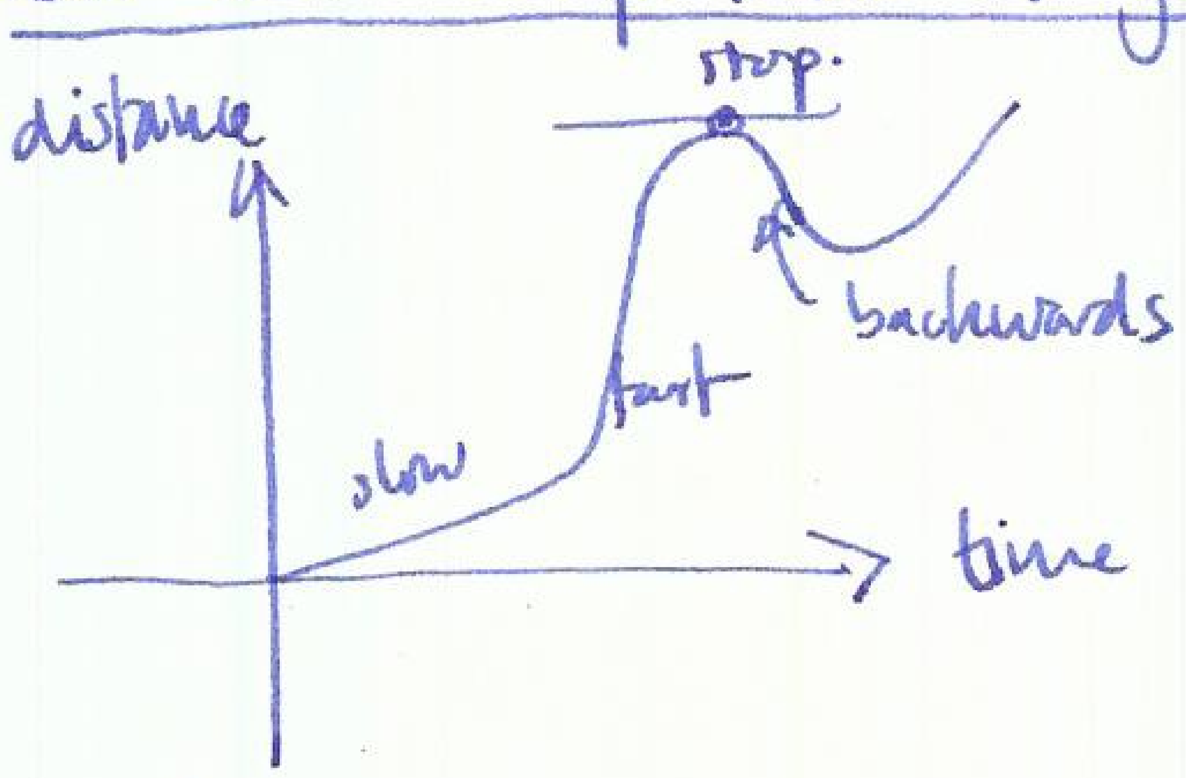
$F'(s) = 1.1 + 0.1s$. $\quad F'(30) = 1.1 + 0.1 . 30 = 4.1$ feet/mph

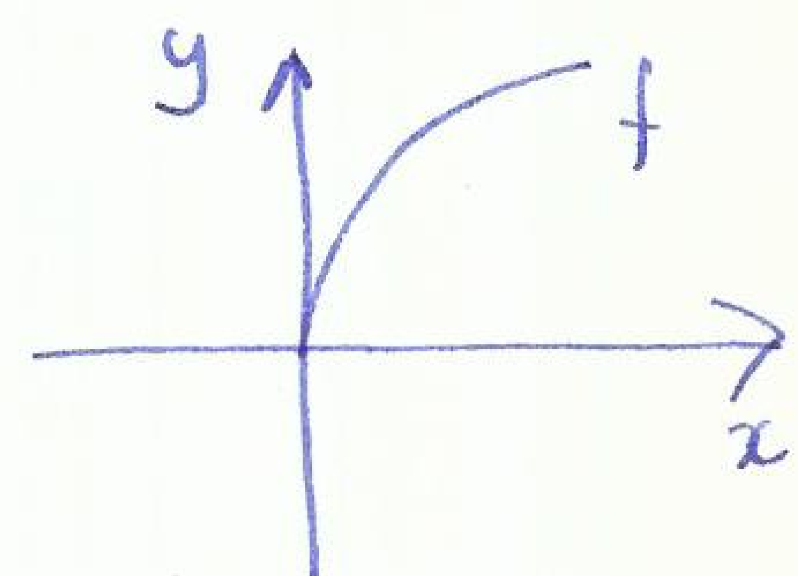
estimate stopping distance at $s = 31$: (using above info).

$$F(s+h) \approx F(s) + hF'(s)$$

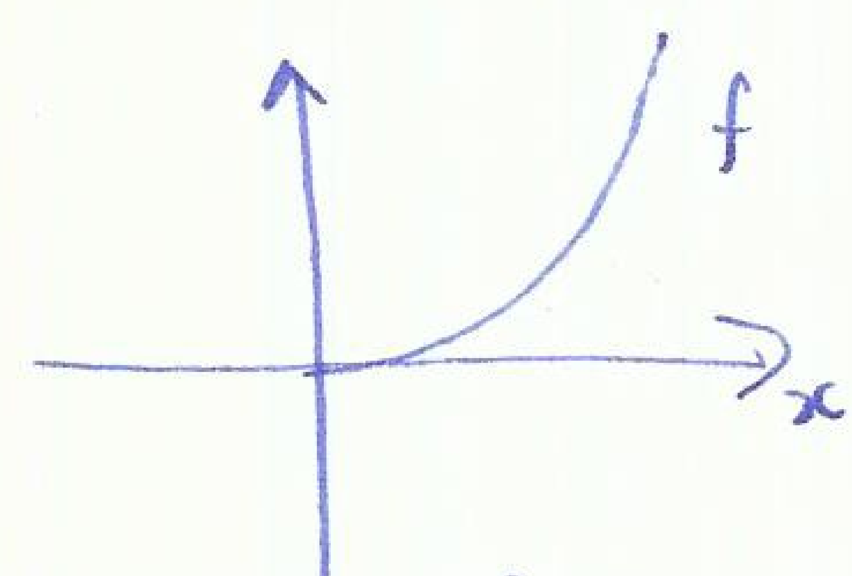
$$F(31) \approx F(30) + 1 . F'(30) = 78 + 1 . 4.1 = 82.1$$

<u>On the interpretation of graphs</u>

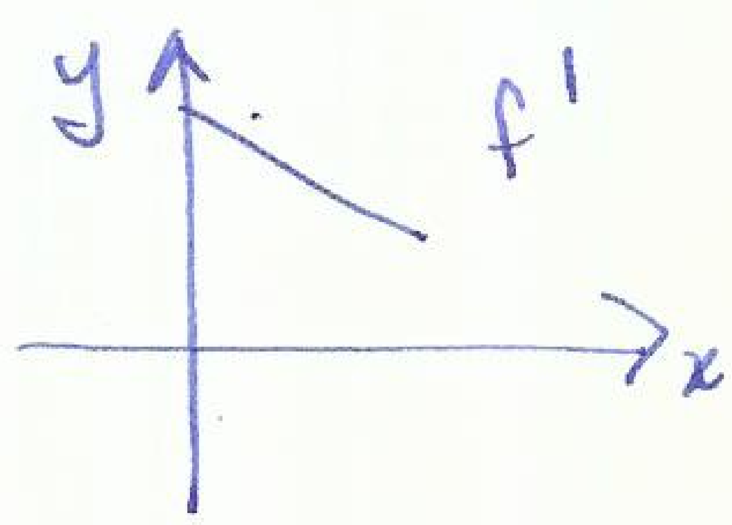


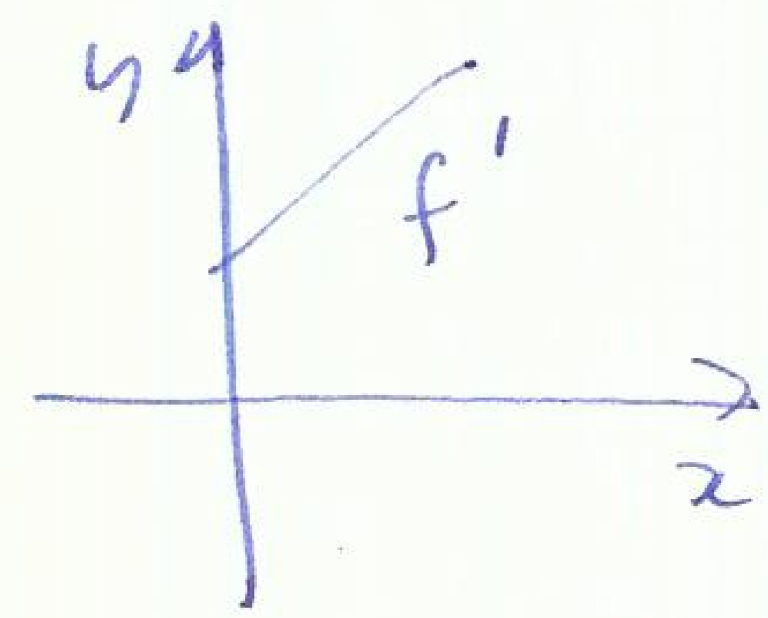


rate of change <u>decreasing</u>

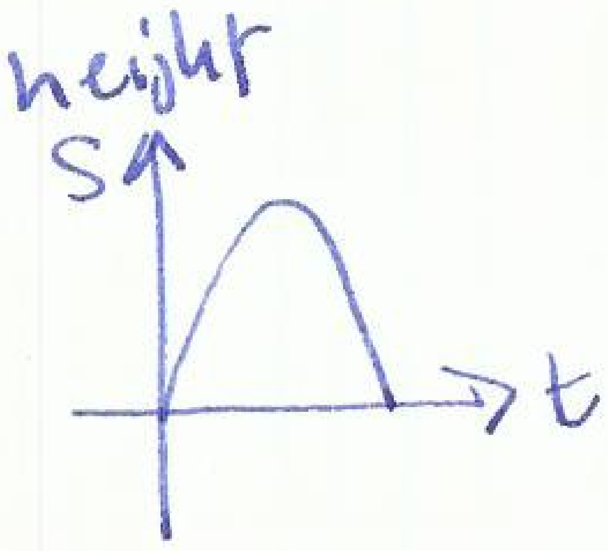


rate of change <u>increasing</u>





<u>Motion under gravity</u>



$$s(t) = s_0 + v_0 t - \tfrac{1}{2} g t^2$$

$$s'(t) = v(t) = v_0 - gt$$

$$s''(t) = a(t) = -g \quad \text{(constant)} \quad g = 9.8\,m/s^2 = 32\,ft/s^2$$

$s_0 = s(0) =$ height at $t=0$ $\qquad v_0 = v(0) =$ speed at $t=0$

Q: when is the maximum height.

A: when $v^{*}(t)=0$: $\quad v_0 - gt = 0 \Rightarrow t = v_0/g$

Example: throw a stone vertically upwards at 10m/s from a height of 2m. what is the maximum height?

$$s(t) = 2 + 10t - \tfrac{1}{2}gt^2$$

$$v(t) = 10 - gt \qquad v(t)=0 \Rightarrow \quad t = \frac{10}{g} \approx 1 \qquad s(1) = 2+10-5 = 7m$$

Q: If I can throw a stone 10m high how fast can I throw it?