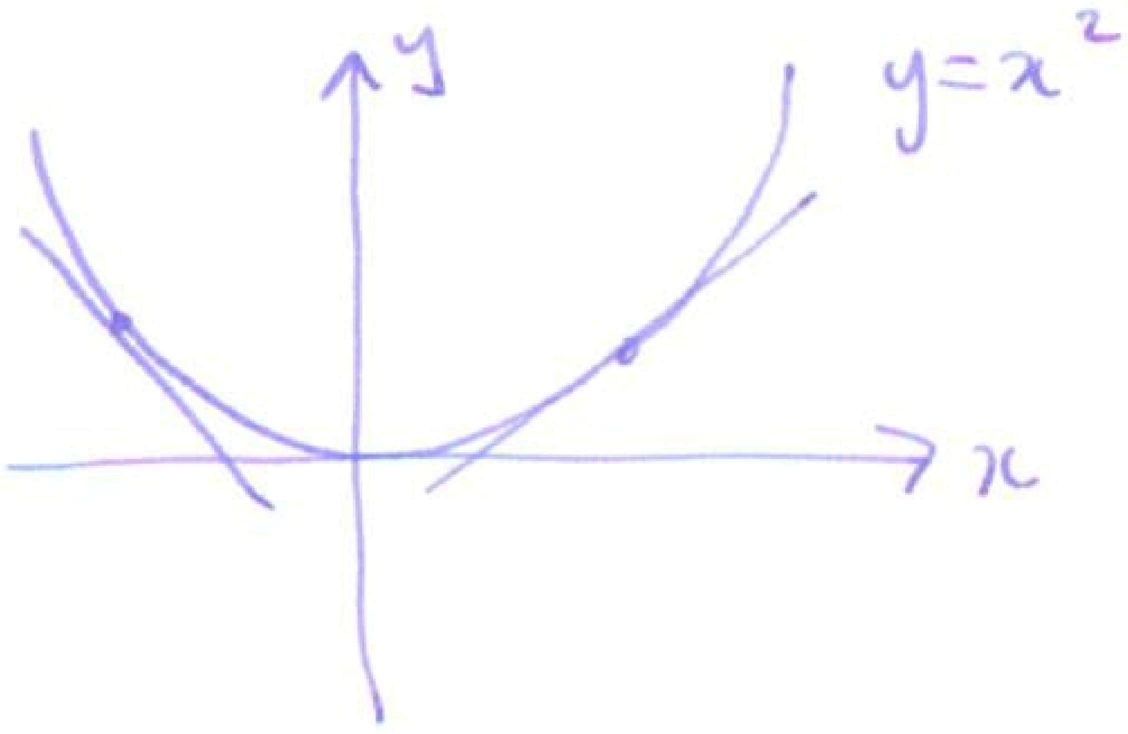
§3.2 Derivative as a function



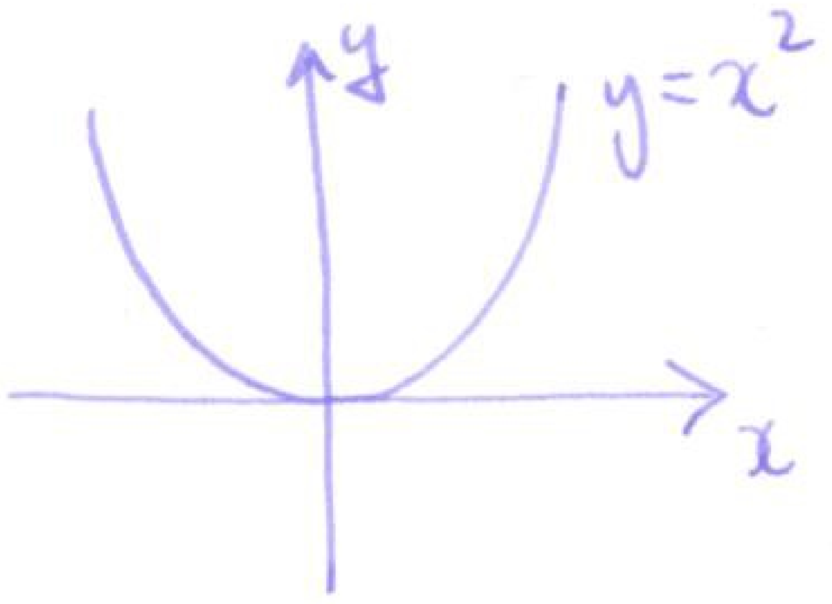
$f : \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

at each (input) point x; there is a tangent line with a slope, which is a number.
Therefore we can define a function

$x \longmapsto$ slope of tangent line at x

<u>notation</u>: we call this function $f'(x)$
or "the derivative of f"

<u>Example</u>



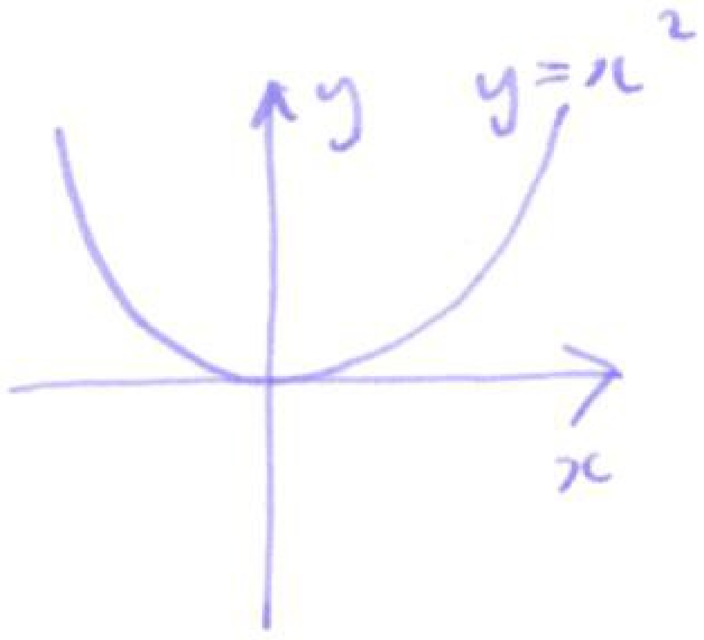
slope at x:

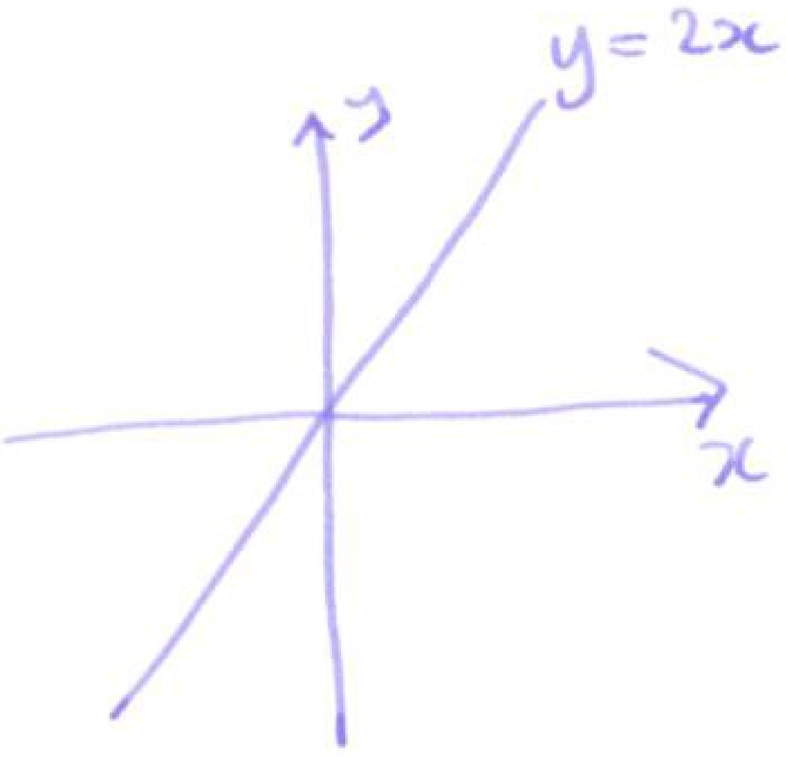
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if $f(x) = x^2$ then: $$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

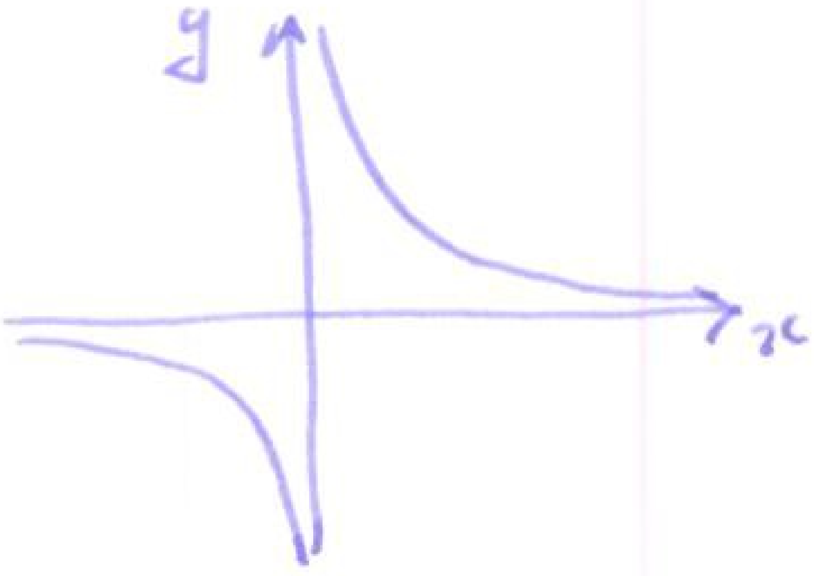
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x.$$

<u>summary</u> if $f(x) = x^2$, then $f'(x) = 2x$





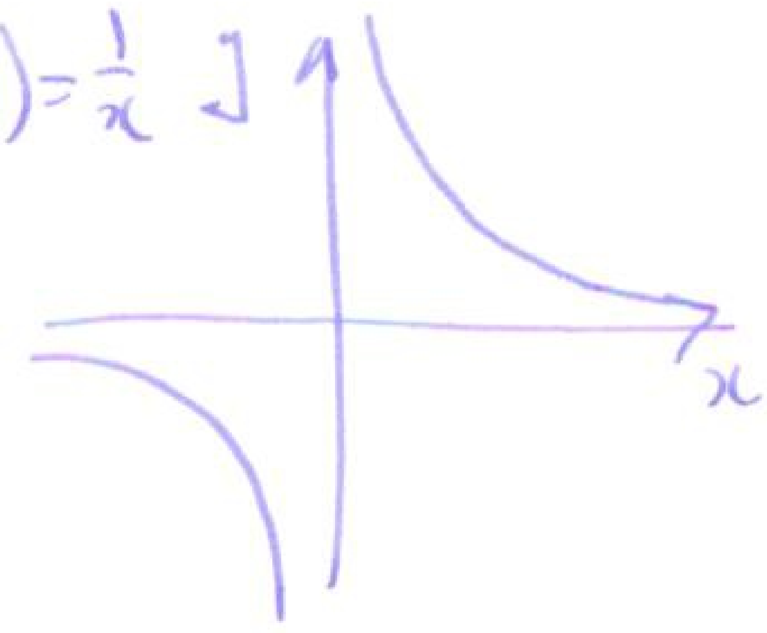
<u>Example</u> $f(x) = \frac{1}{x}$



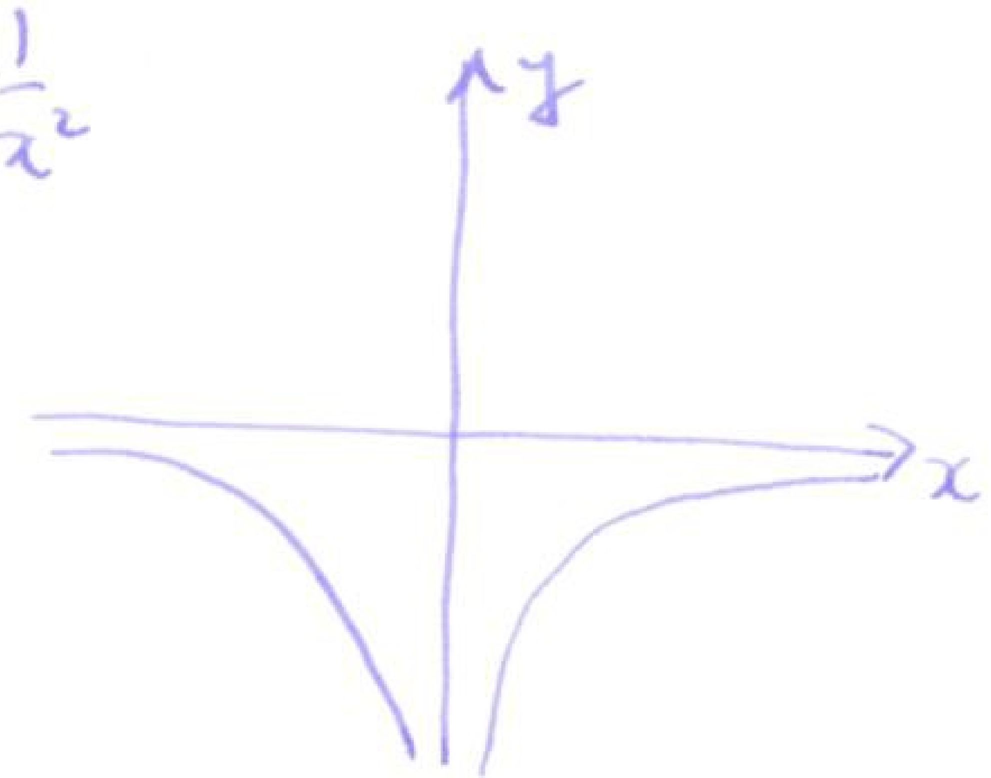
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\, \frac{x-(x+h)}{(x+h)\,x}$$

$$= \lim_{h\to 0} \frac{-h}{h\,x(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

$f(x)=\frac{1}{x}$



$f'(x) = -\frac{1}{x^2}$



Remarks

① functions : f : domain $\longrightarrow$ range.

Warning : not all functions differentiable

e.g. $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$\underset{f}{\{\text{differentiable! functions}\}} \xrightarrow{\text{derivative}} \underset{f' \text{ or } \frac{df}{dx}}{\{\text{functions}\}}$$

② "calculus" means rules for doing calculations – we won't have to compute explicit limits all the time.

Example $f(x) = x^3$ $\quad f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Thm (powers of x) if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Example $\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2 \qquad \frac{d}{dx}(x^0) = 0 . x^{-1} = 0$

$\frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$ $\quad$ note $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

Proof let $f(x) = x^n$

then $f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2} x^{n-2} h^2 + \cdots + h^n}_{\text{all terms contain at least } h^2}$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and where $n! = 1.2.3 \ldots n$.

so $\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \cdots + h^n}{h}$

$= \lim_{h \to 0} nx^{n-1} + \binom{n}{2} x^{n-2} h + \cdots + h^{n-1} = nx^{n-1}$ $\square$.

Warning: this rule works for polynomials only, not exponentials.

$f(x) = x^2$ power of x. (e.g. $x^{\sqrt{2}}, x^{\pi}$)

$f(x) = 2^x$ not a power of x.

Other useful rules

Thm (linearity) If f and g are differentiable, then $f+g$ is differentiable and $(f+g)' = f' + g'$

$\Leftrightarrow \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

also if c is a constant $(cf)' = cf'$

$\Leftrightarrow \frac{d}{dx}(cf) = c\frac{df}{dx}$

Proof (from limit laws)

$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x)$$

constant multiple: $\displaystyle \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} = c \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = c f'(x)$ □

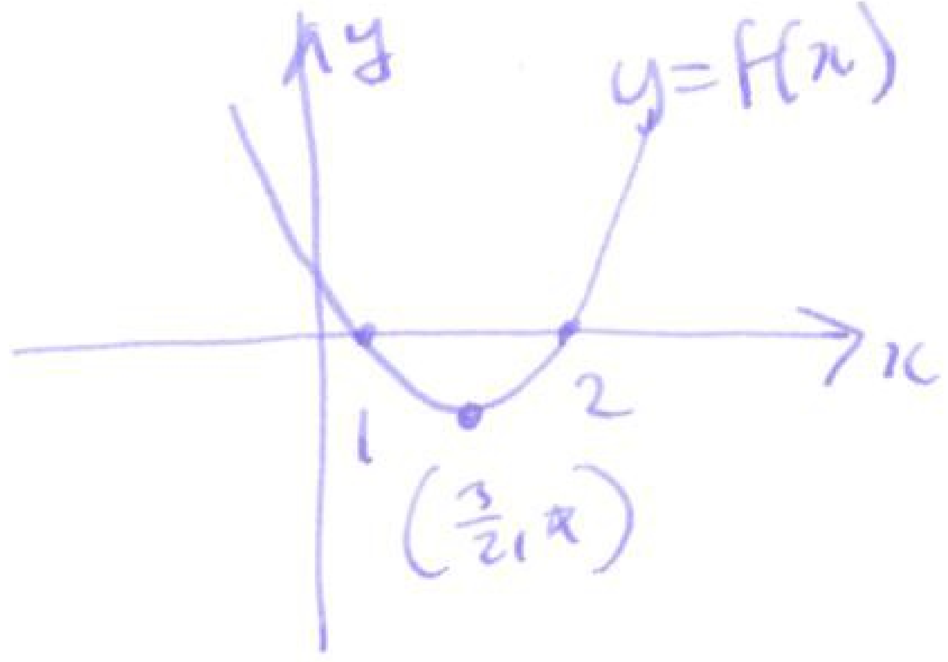
Example $f(x) = x^2 - 3x + 2$ find $f'(x)$

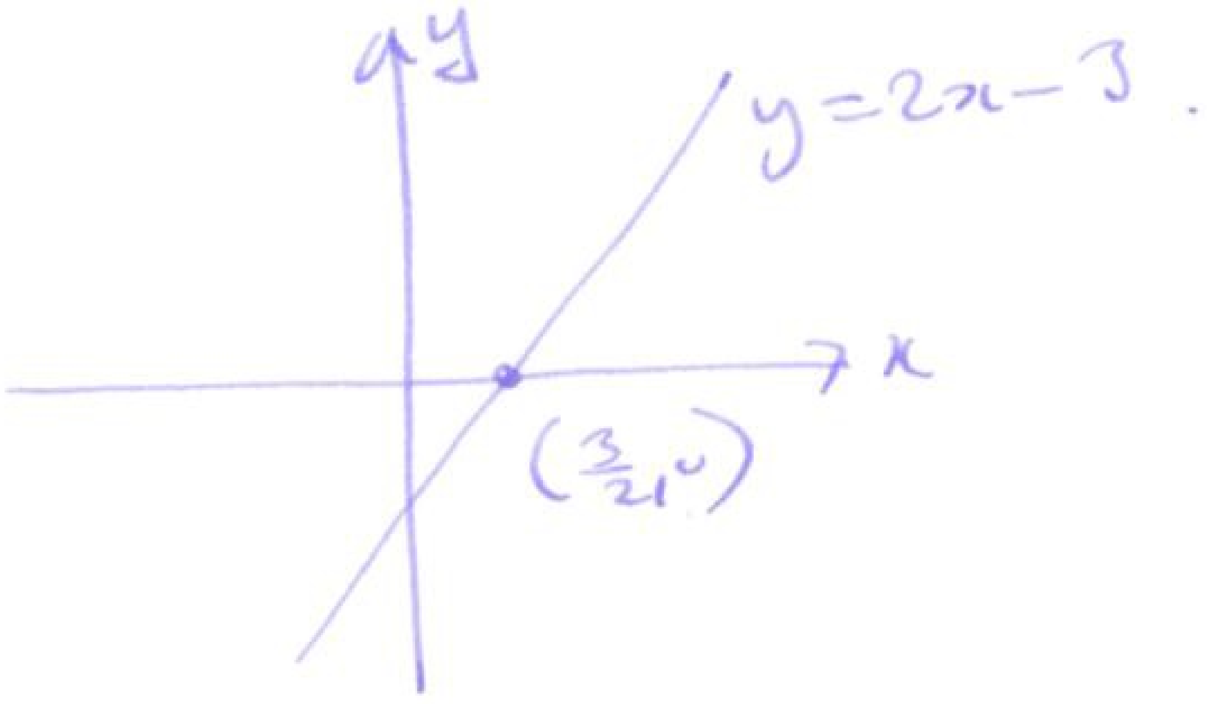
$$\frac{df}{dx} = \frac{d}{dx}(x^2-3x+2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2) \quad \text{(sums)}.$$

$$= \frac{d}{dx}(x^2) - 3\frac{d}{dx}(x) + \frac{d}{dx}(2) \quad \text{(constant multiple)}$$

$$= 2x - 3 + 0 \quad \text{(power rule)}.$$

graphs $f(x) = x^2 - 3x + 2 = (x-2)(x-1)$





Derivative of e^x

more generally, consider $f(x) = b^x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} = \lim_{h\to 0} b^x \frac{b^h-1}{h}$$

$$= b^x \underbrace{\lim_{h\to 0} \frac{b^h-1}{h}}$$ doesn't depend on x!
assume this limit exists and call it m_b

we have shown: for exponential functions: derivative is proportional to function

if $f(x) = b^x$ then $f'(x) = m_b b^x$