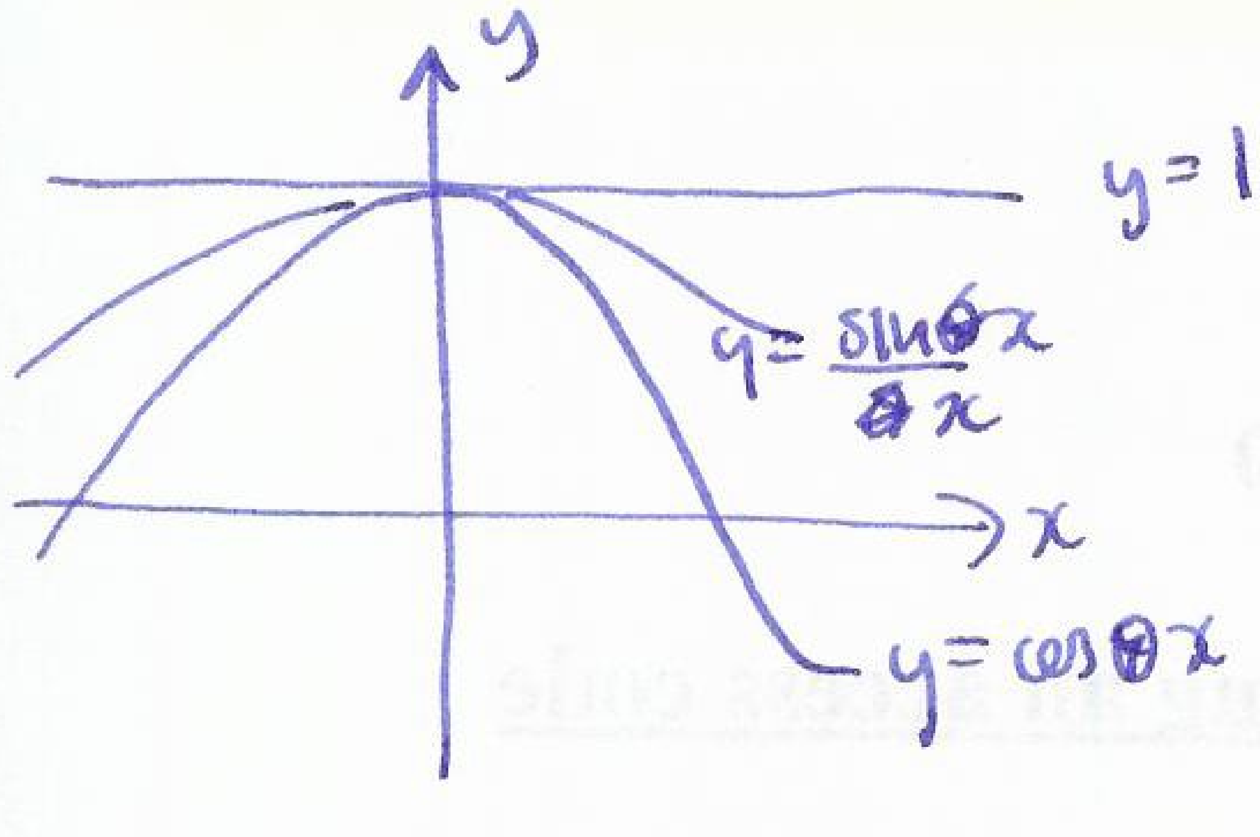


$$\left.\begin{array}{l} \lim_{\theta\to 0} \cos\theta = 1 \\ \lim_{\theta \to 0} 1 = 1 \end{array}\right\} \Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

substitute: ($\cos\theta$ continuous)

Example

① $\lim_{x\to 0} \frac{\sin 3x}{2x}$ Know: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

write this as: $\theta = 3x \iff x = \frac{\theta}{3}$.

$$\lim_{\theta\to 0} \frac{\sin\theta}{2\theta/3} = \lim_{\theta\to 0} \frac{3}{2}\frac{\sin\theta}{\theta} = \frac{3}{2} \underbrace{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}_{=1} = \frac{3}{2}.$$

②

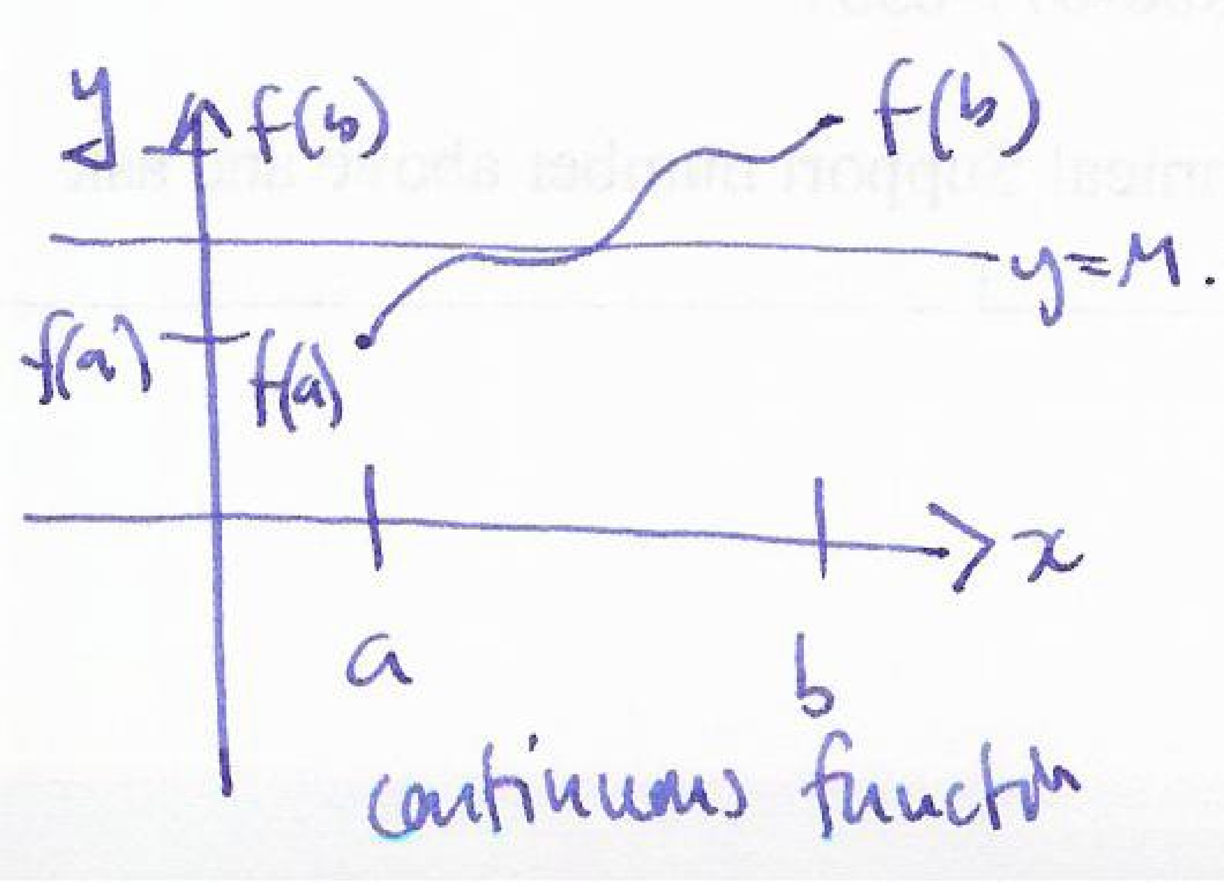
$$\lim_{t\to 0} \frac{1-\cos t}{\sin t} = \lim_{t\to 0} \frac{1-\cos t}{t}\cdot\frac{t}{\sin t}$$

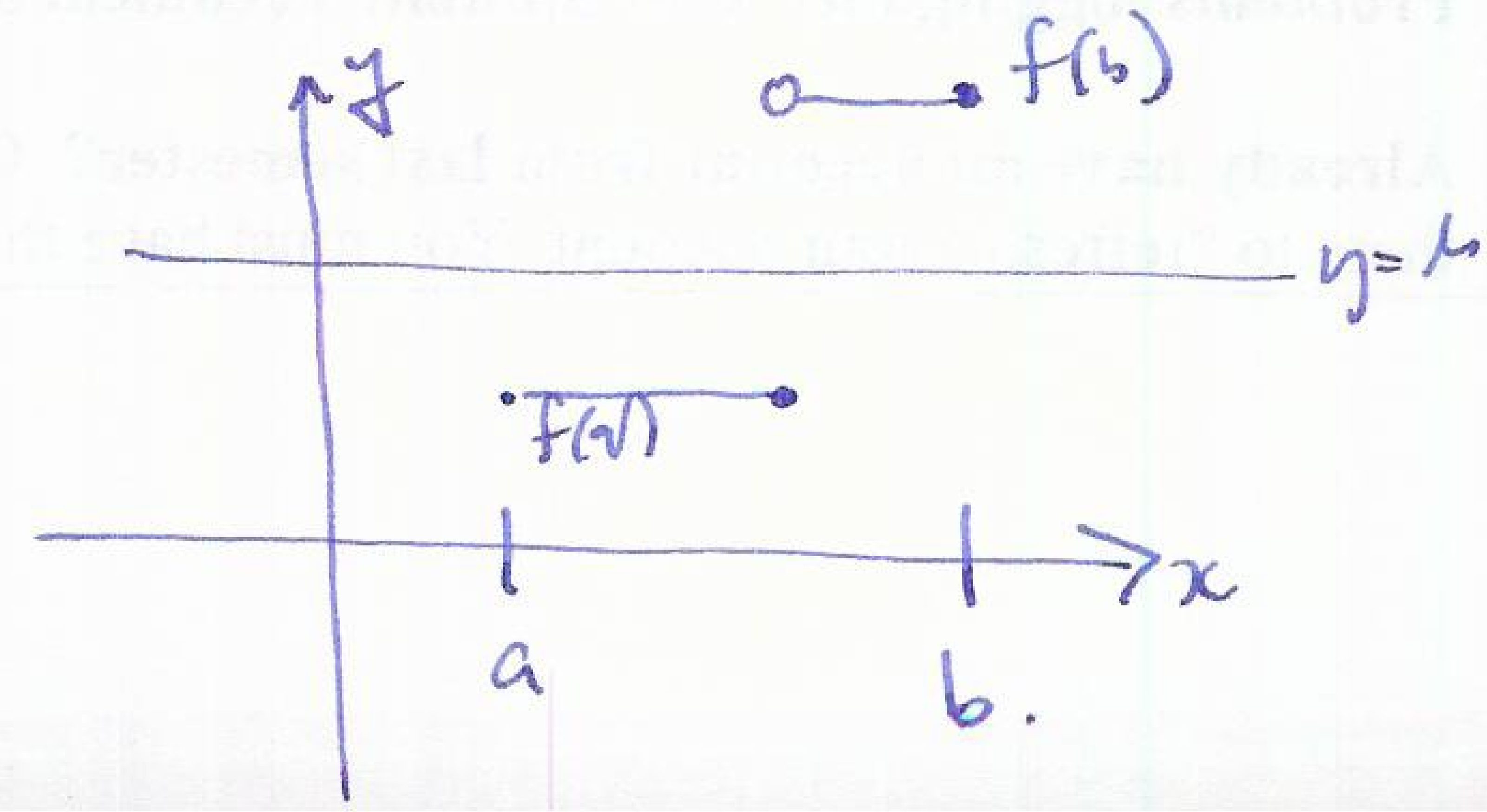
note: $\lim_{t\to 0} \frac{1-\cos t}{t} = 0$ $\quad \lim_{t\to 0}\frac{t}{\sin t} = \lim_{t\to 0} \frac{1}{\frac{\sin t}{t}} = \frac{1}{\lim_{t\to 0}\frac{\sin t}{t}} = 1.$

so $\lim_{t\to 0} \frac{1-\cos t}{\sin t} = 0\cdot 1 = 0.$

§2.7 Intermediate value theorem

"continuous functions can't skip values".



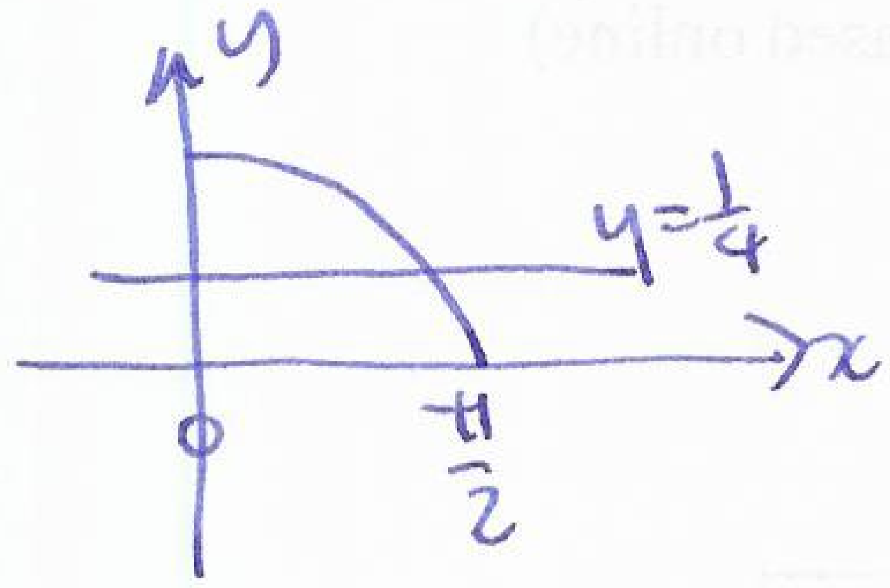


Thm (Intermediate Value Theorem IVT)

If $f(x)$ is a continuous function on a closed interval $[a,b]$ and $f(a) \neq f(b)$. Then for every M between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ such that $f(c) = M$.

Application: showing that equations have solutions.

Example show $\cos(x) = \frac{1}{4}$ has at least one solution.



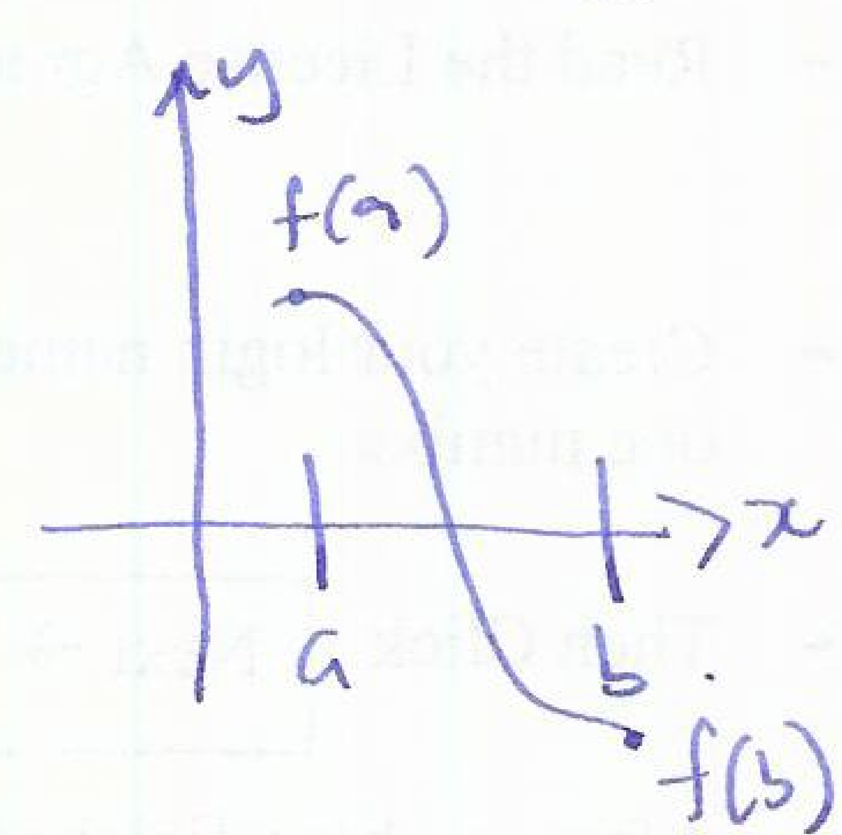
consider $[0, \frac{\pi}{2}]$: $\cos(0) = 1$

$\cos(\frac{\pi}{2}) = 0$

$0 \leq \frac{1}{4} \leq 1 \Rightarrow \cos(x) = \frac{1}{4}$ has at least one solution.

special case: finding zeros:

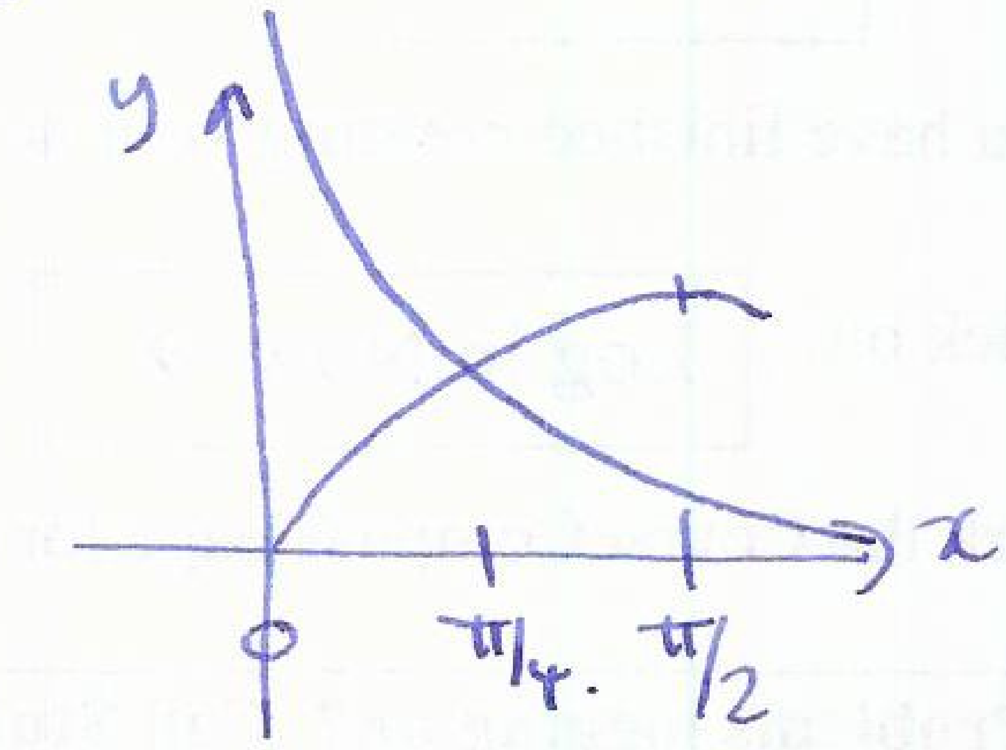
Corollary If $f(x)$ is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have different signs, then there is at least one $c \in [a,b]$ such that $f(c) = 0$



Bisection method:

find a solution to $\sin x = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

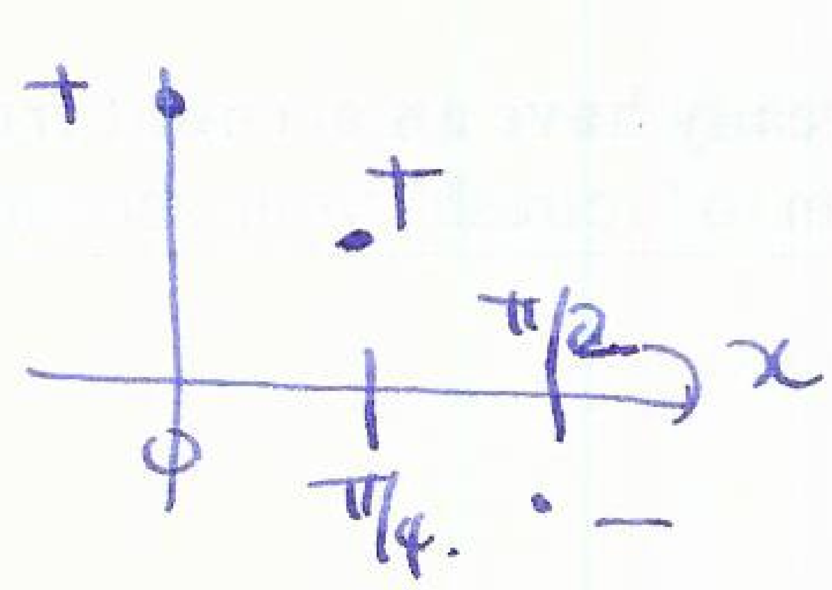
consider $f(x) = \frac{1}{x} - \sin(x)$



$f(0) = +\infty$

$f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 \approx -0.3634...$

now try halfway point $f(\frac{\pi}{4})$



$f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.566$

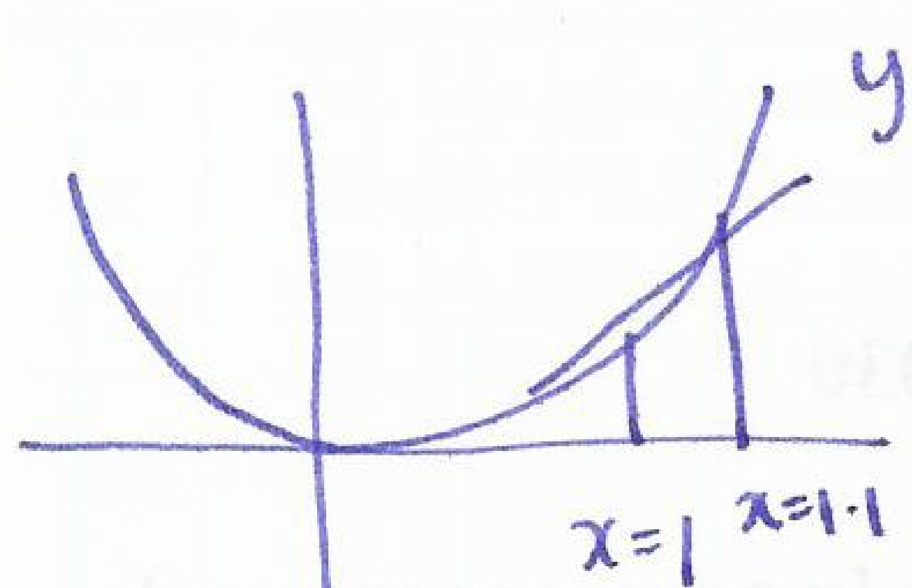
so choose $[\frac{\pi}{4}, \frac{\pi}{2}]$ ← so there is a solution in this interval.

$f(\frac{\pi}{4}) > 0$ $f(\frac{\pi}{2}) < 0$

now try midpoint $\frac{3\pi}{8}$.....

§3.1 Definition of the derivative

<u>recall</u>:



we can compute the average rate of change of a function over some interval $[x_0, x_1]$ i.e. $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

<u>Q</u>: how do compute the slope of the tangent line?

<u>idea</u>: look at average rate of change over small interval $[x, x+h]$ and take limit at $h \to 0$.

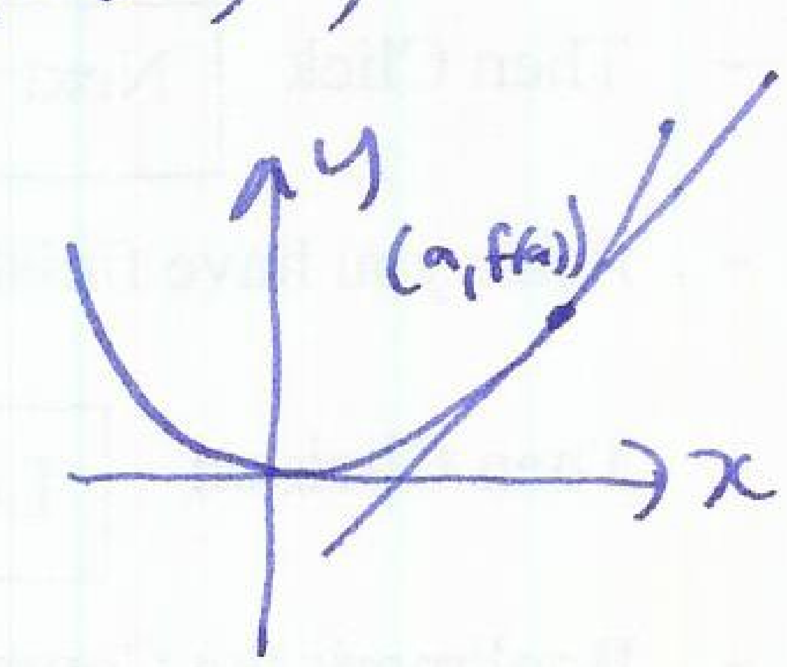
<u>Def$^{\underline{n}}$</u> the slope of the tangent line at $x=a$ is $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

<u>notation</u> also called <u>the derivative</u> written $f'(a)$ (Newton) $\dfrac{df}{dx}(a)$ (Liebnitz).

if this limit exists we say f is <u>differentiable</u> at $x=a$.

<u>note</u> $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ same as $\displaystyle\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$.

<u>Def$^{\underline{n}}$</u> The tangent line to $f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$



the equation for this line is $y - y_0 = m(x - x_0)$

i.e. $y - f(a) = f'(a)(x - a)$

<u>Example</u> find tangent line to $y = x^2$ at $x = 1$

point: $(1,1)$ slope: $f'(1) = \displaystyle\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$

$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} 2+h = 2$$

so equation of tangent line is $y-1 = 2(x-1)$

Example find slope of tangent line to $y=\frac{1}{x}$ at $x=4$

point: $(4,\frac{1}{4})$ slope: $f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h}-\frac{1}{4}}{h}$

$$= \lim_{h\to 0} \frac{1}{h} \frac{4-(4+h)}{4(4+h)} = \lim_{h\to 0} \frac{4-4-h}{4h(4+h)} = \lim_{h\to 0} \frac{-h}{4h(4+h)}$$

$$= \lim_{h\to 0} \frac{-1}{4(4+h)} = \frac{-1}{16}$$

Example straight line $y=mx+b$.

slope when $x=a$: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{m(a+h)+b-ma-b}{h}$

$$= \lim_{h\to 0} \frac{\cancel{ma}+mh+\cancel{b}-\cancel{ma}-\cancel{b}}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m.$$

Observation if $f(x)=b$ (constant) then $f'(a)=0$ for all a.