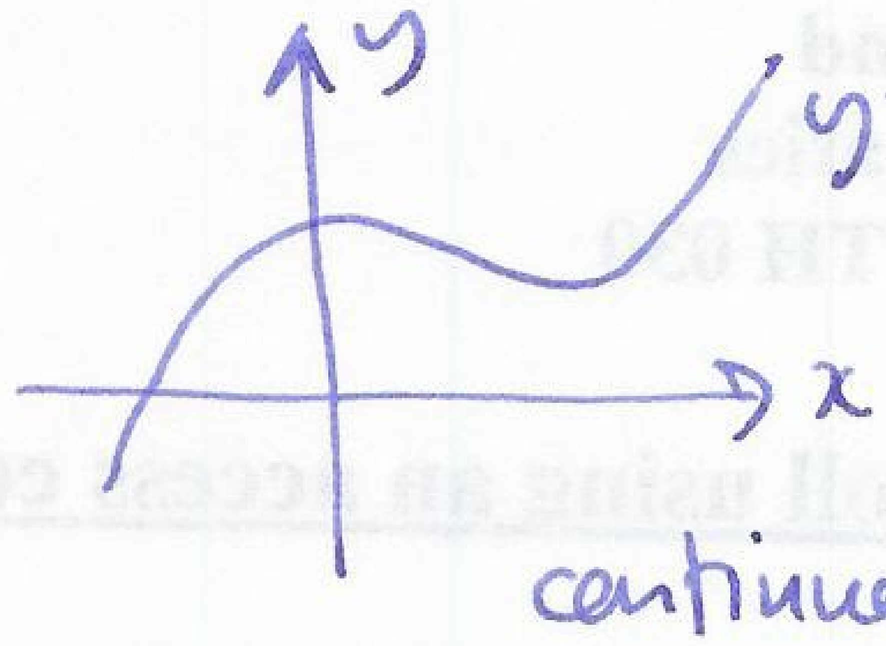
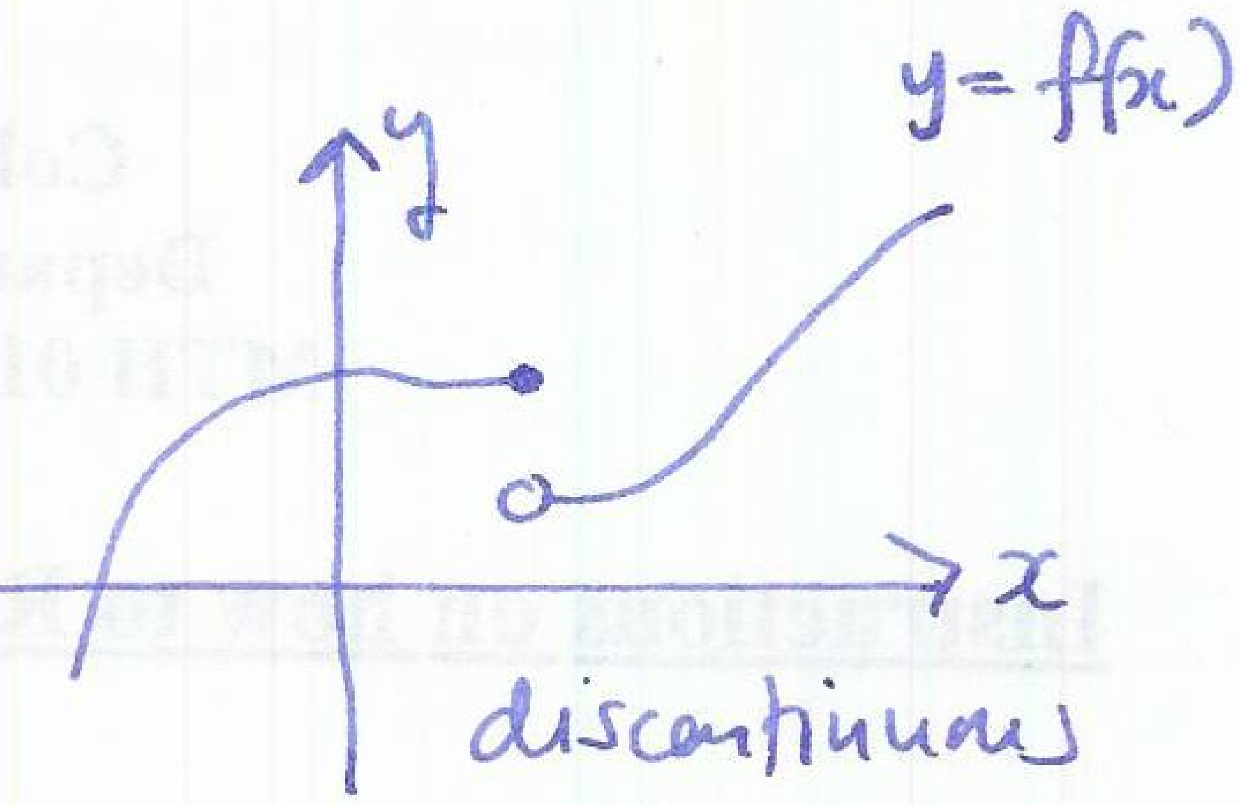
§2.4 Limits and continuity

Example



continuous

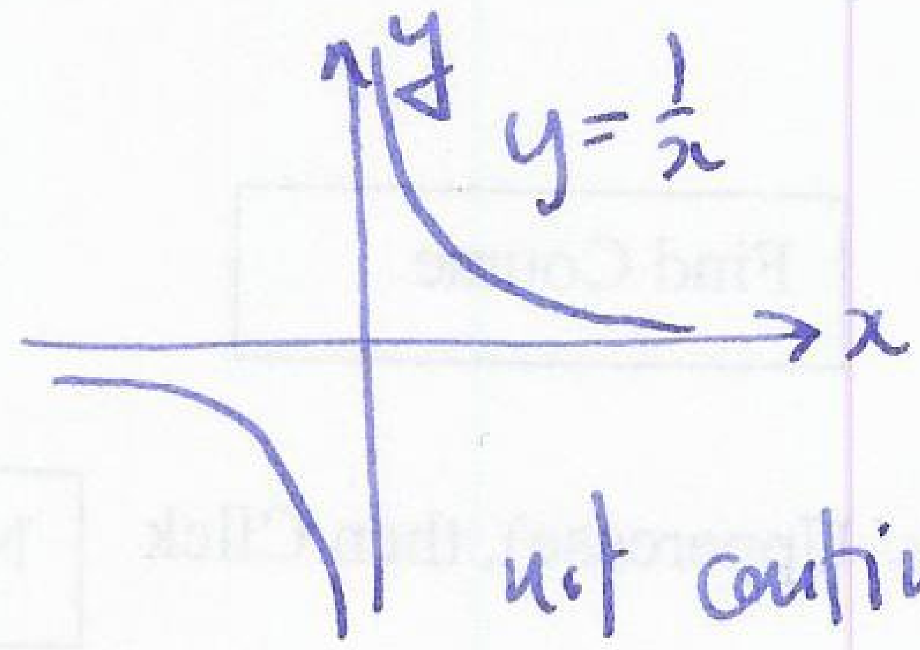


discontinuous

Def^n^ we say $f(x)$ is continuous at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

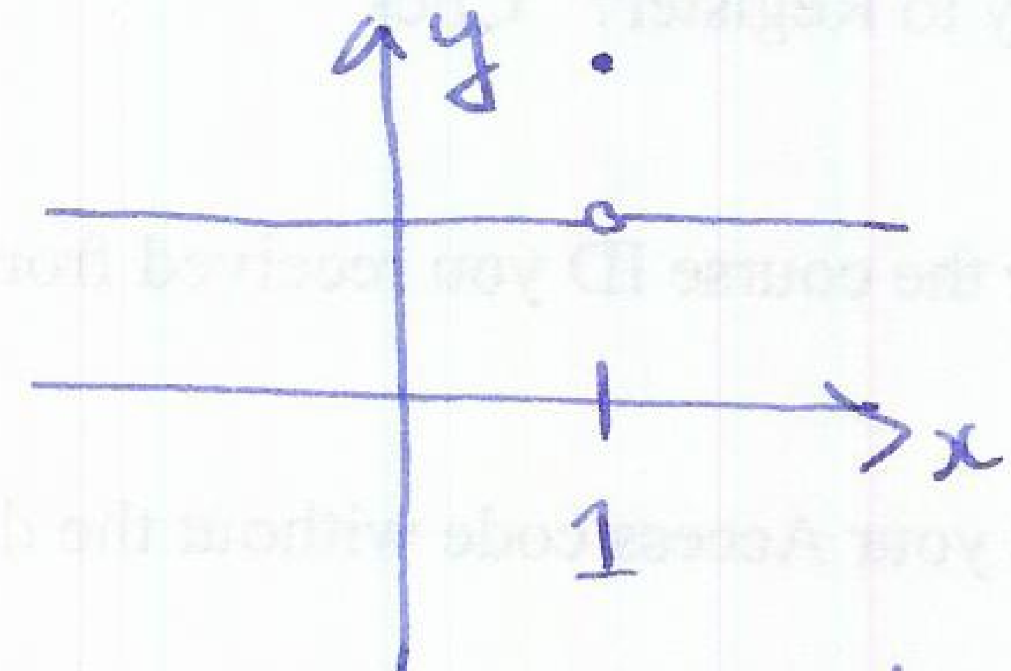
If the limit does not exist, or is not equal to $f(c)$, then $f(x)$ is not continuous (discontinuous) at $x=c$.

Example



not continuous at $x=0$

$f(x)=\frac{1}{x}$.

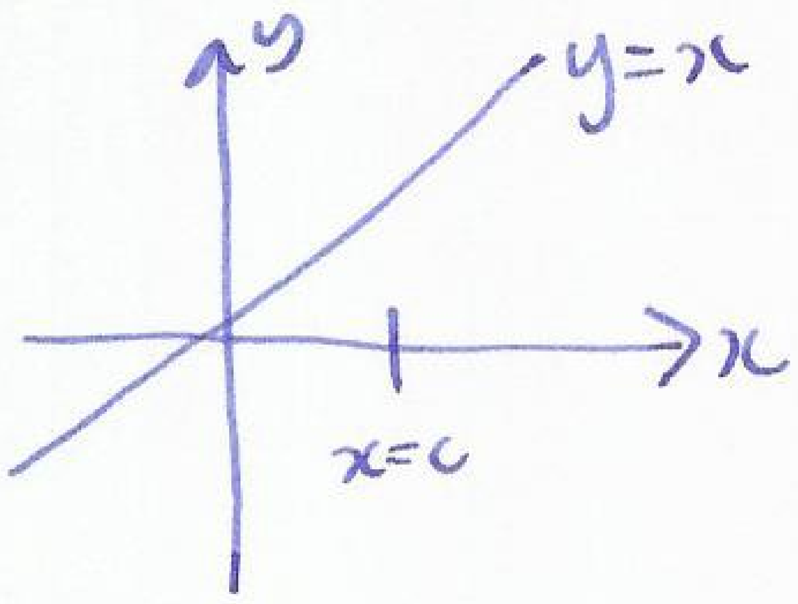


not continuous at $x=1$.

$$f(x) = \begin{cases} 1 & x \neq 2 \\ 2 & x = 1 \end{cases}$$

Example

show $f(x)=x$ is continuous



$f(c)=c$

want to show: $\lim_{x\to c} f(x) = c$

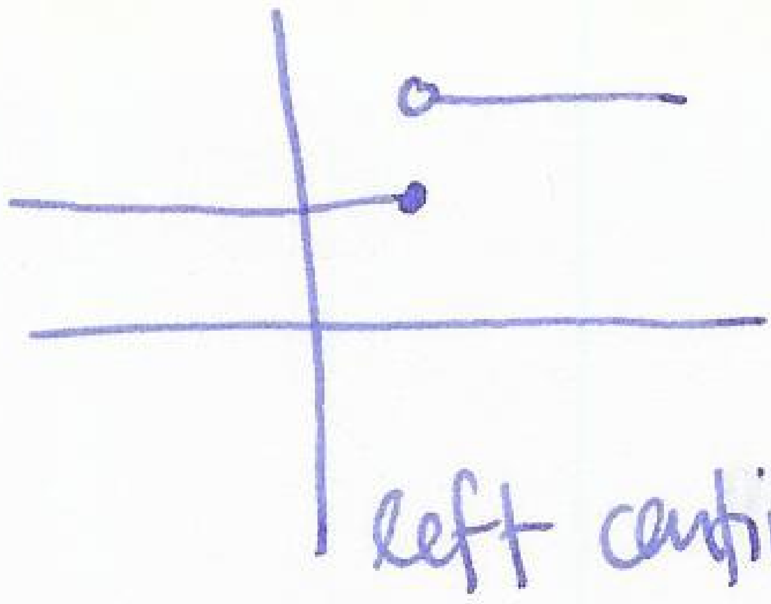
Proof $|f(x)-L| = |x-c|$ get small as $x\to c$ $\square$

(already did this one)

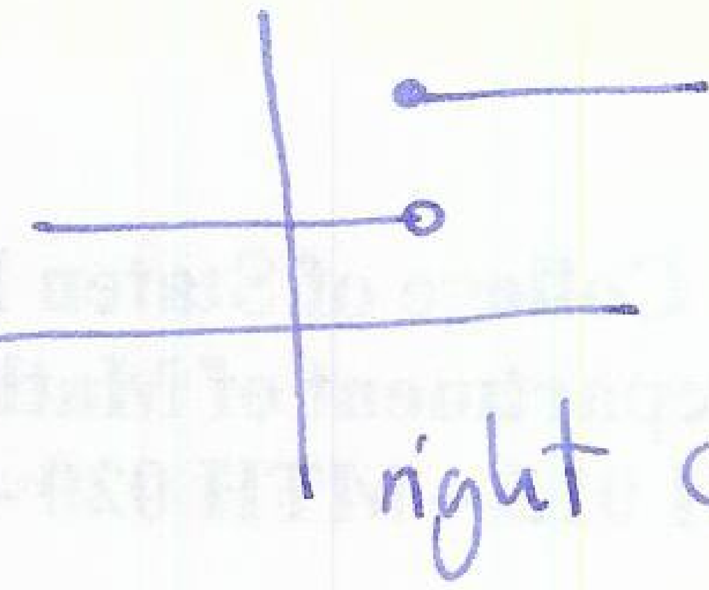
Def^n^ $f(x)$ is left continuous at $x=c$ if $\lim_{x\to c-} f(x) = f(c)$

right continuous at $x=c$ if $\lim_{x\to c+} f(x) = f(c)$

Examples



left continuous



right continuous

If at least one of the left or right limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x=c$.

<u>Building continuous functions</u>

<u>Thm</u> 1 Suppose that $f(x)$ and $g(x)$ are both continuous at $x=c$, then the following functions are continuous at $x=c$.

1) $f(x)+g(x)$

2) $kf(x)$ for any constant k

3) $f(x)g(x)$

4) $\dfrac{f(x)}{g(x)}$ if $g(c)\neq 0$

<u>Proof</u>: these follow directly from the limit laws.

<u>e.g.</u> if $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist then

$$\lim_{x\to c}\left(f(x)+g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$, which gives 1)

etc $\square$.

<u>Thm</u> 2 Polynomials $p(x)$ are continuous

Rational functions $\dfrac{p(x)}{q(x)}$ are continuous, except where $q(x)=0$.

Proof $f(x)=x$ is continuous

so $f(x)f(x) = x^2$ is continuous $\Rightarrow x^3, x^4, \dots$ continuous.

so $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ is continuous.

so $\frac{p(x)}{q(x)}$ is continuous as long as $q(x) \neq 0$ $\square$

Useful facts

Thm 3
- $\sin(x)$, $\cos(x)$ are continuous
- b^x is continuous $(b>0)$
- $\log_b(x)$ is continuous
- $x^{1/n}$ is continuous

(inverse functions)

Thm 4 If $f: D \to R$ is continuous, with inverse $f^{-1}: R \to D$, then f^{-1} is continuous.

Thm 5 (composition) If $f(x)$ is continuous at $x=c$ and $g(x)$ is continuous at $x=f(c)$

then $g(f(x))$ is continuous at $x=c$.

Example $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2+x+1}}$ continuous at $x=1$

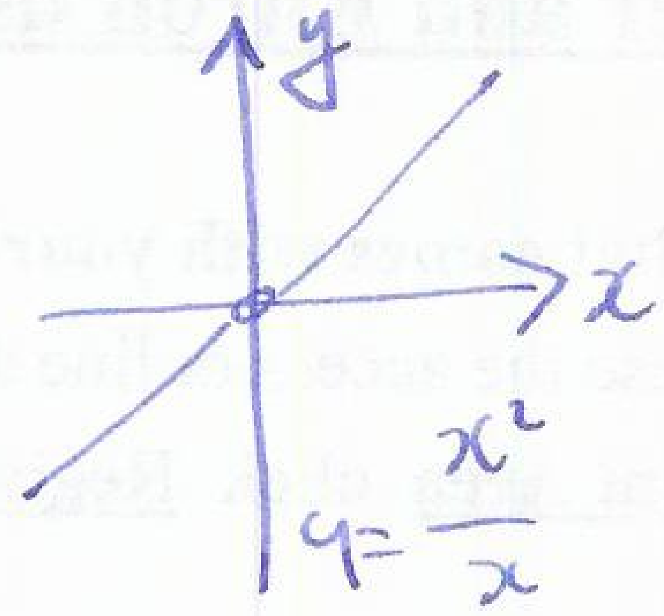
Q: when is $f(x) = \dfrac{x^2}{\sin(x)}$ cts?

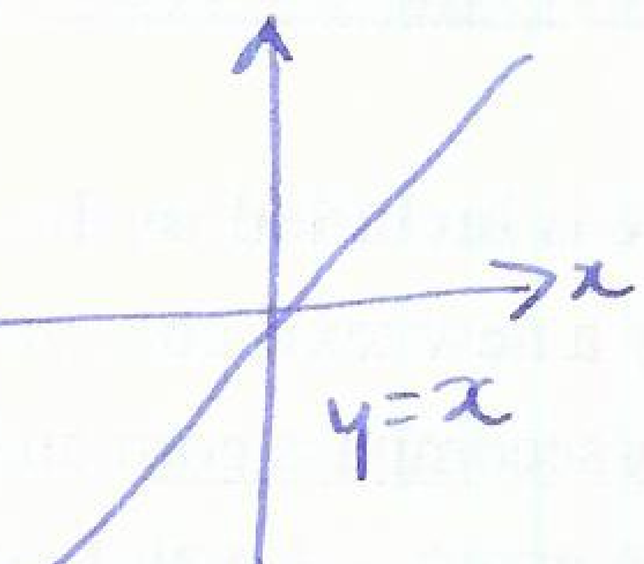
§2.5 Evaluating limits algebraically

Example $\lim_{x\to 0} \frac{x^2}{x}$ undefined at $x=0$: $\frac{0}{0}$ indeterminate form.

but: limit does not depend on value of function at $x=0$.

$\frac{x^2}{x} = x$ for $x \neq 0$





so $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$.

Indeterminate forms : $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty . 0$, $\infty - \infty$

note : $\frac{1}{0}$ not indeterminate, gives $\pm\infty$ or DNE.

Example $\lim_{x\to 3} \frac{x^2-4x+3}{x^2+x-12}$ $\qquad \frac{9-12+3}{9+3-12} = \frac{0}{0}$.

factor : $\frac{(x-3)(x-1)}{(x-3)(x+4)} = \frac{x-1}{x+4}$ $(x\neq 3)$

so $\lim_{x\to 3} \frac{x^2-4x+3}{x^2+x-12} = \lim_{x\to 3} \frac{x-1}{x+4} = \frac{2}{7}$

Example $\lim_{x\to \frac{\pi}{2}} \frac{\tan x}{\sec x}$ $\qquad \tan(\pi/2) = \frac{\sin(\pi/2)}{\cos(\pi/2)} = 1/0$ undefined.

$\sec(\pi/2) = \frac{1}{\cos(\pi/2)} = 1/0$ undefined.

but $\frac{\tan(x)}{\sec(x)} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \sin x$ if $\cos x \neq 0$.

so $\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1.$

<u>Example</u> $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $\quad \frac{2-2}{4-4} = \frac{0}{0}$ undefined.

$$\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2} \quad \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

(or difference of two squares $x-4 = (\sqrt{x}-2)(\sqrt{x}+2)$.).

<u>Example</u> $\lim_{x\to 5} \frac{x-5}{\sqrt{x+4}-3} = 6. \quad \left(\times \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}\right)$

<u>Example</u> $\lim_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1}$ $\quad \frac{1}{0} - \frac{2}{0}$ undefined

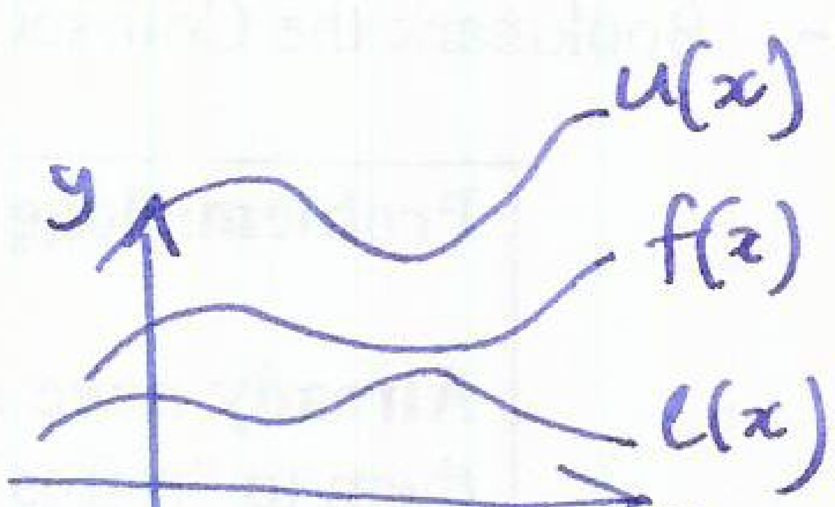
$$\frac{1}{x-1} - \frac{2}{x^2-1} = \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{1}{x+1} \quad \lim_{x\to 1}\frac{1}{x+1} = \frac{1}{2}.$$

<u>§2.6 Trigonometric limits</u>

<u>Q</u>: what is $\lim_{x\to 0} \frac{\sin(x)}{x}$? (substitute: $\frac{0}{0}$ undefined).

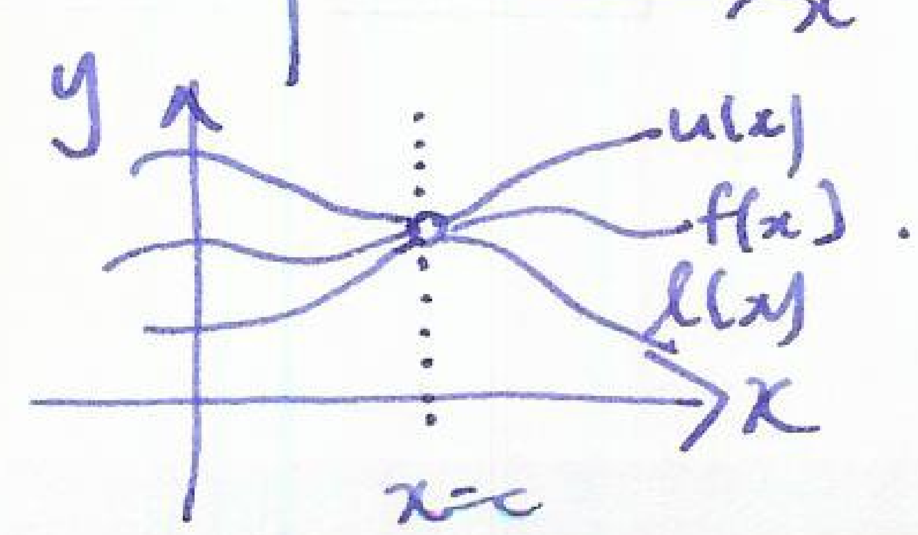
<u>A</u>: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (try: $x = 0.1, 0.01, 0.001, \ldots$)

<u>Squeeze theorem</u>: suppose $l(x) \le f(x) \le u(x)$

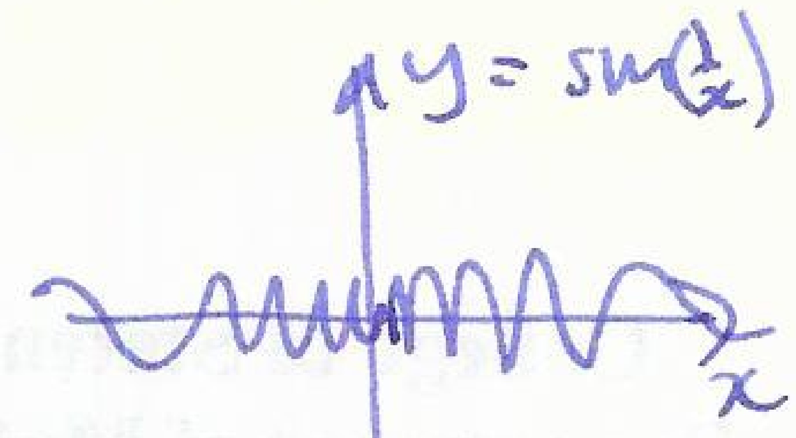


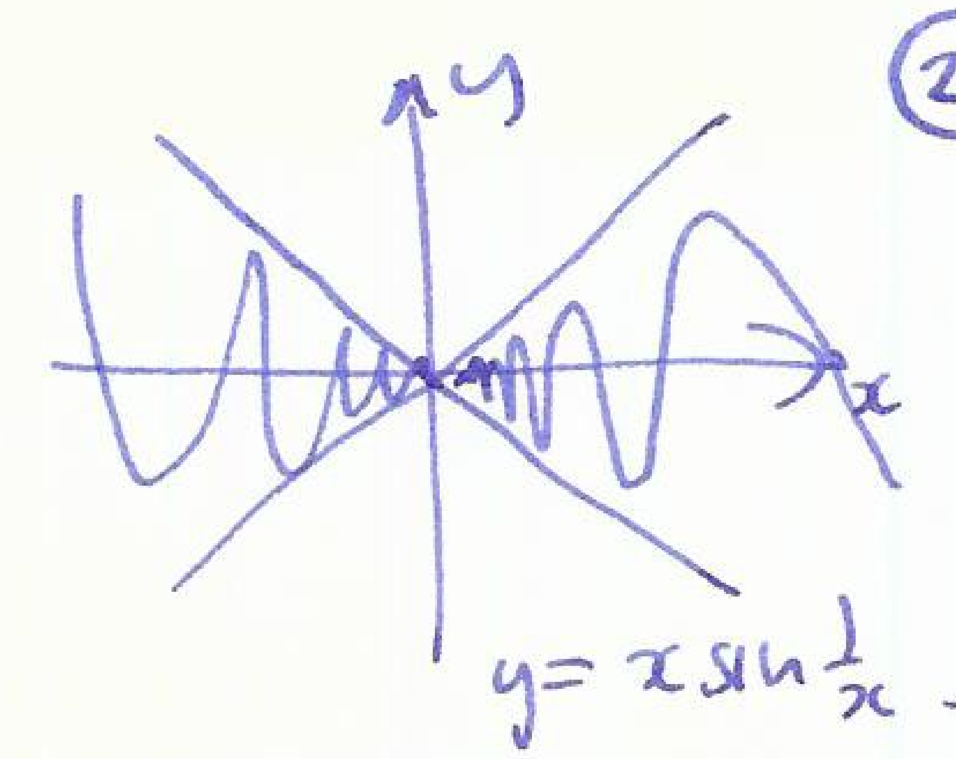
<u>Thm</u> suppose $l(x) \le f(x) \le u(x)$ and

$\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$ then $\lim_{x\to c} f(x) = L$



Example $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$





$$|\sin(\theta)| \le 1$$

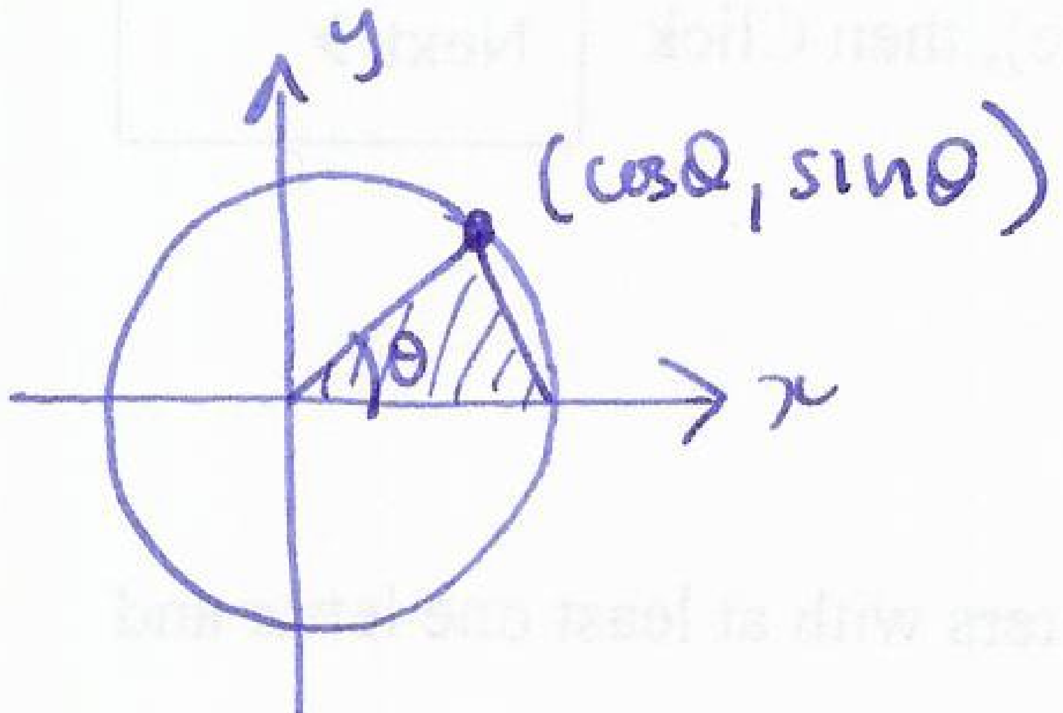
so

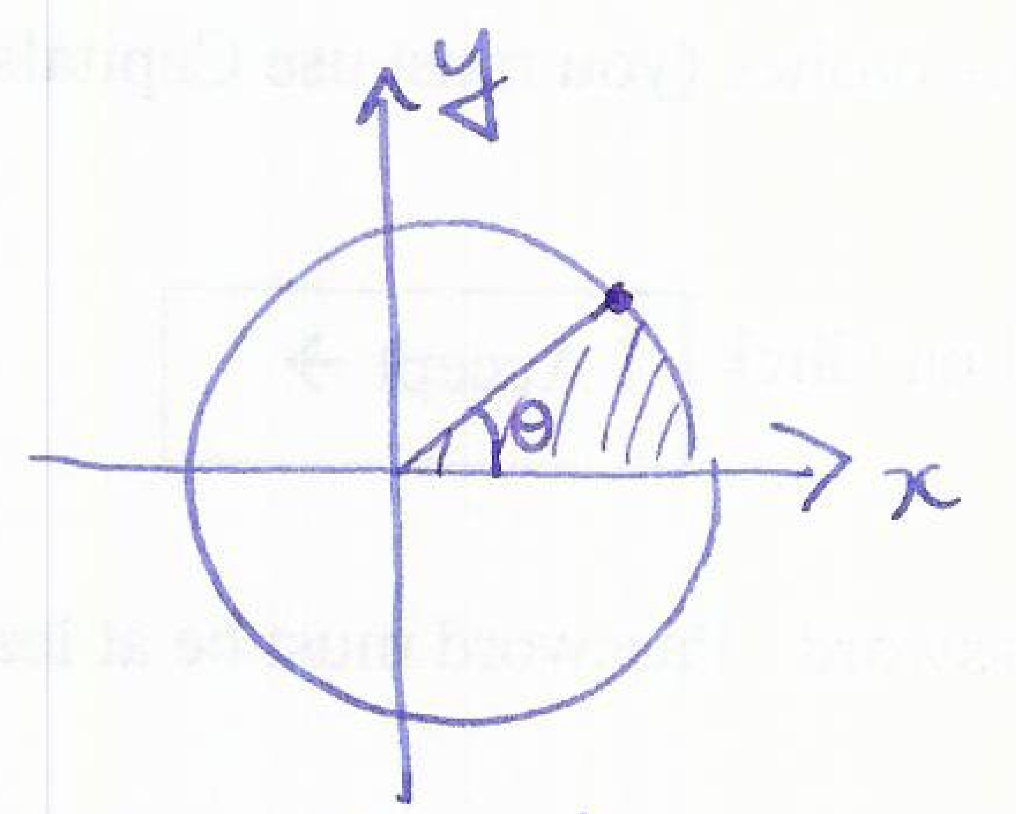
$$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$$

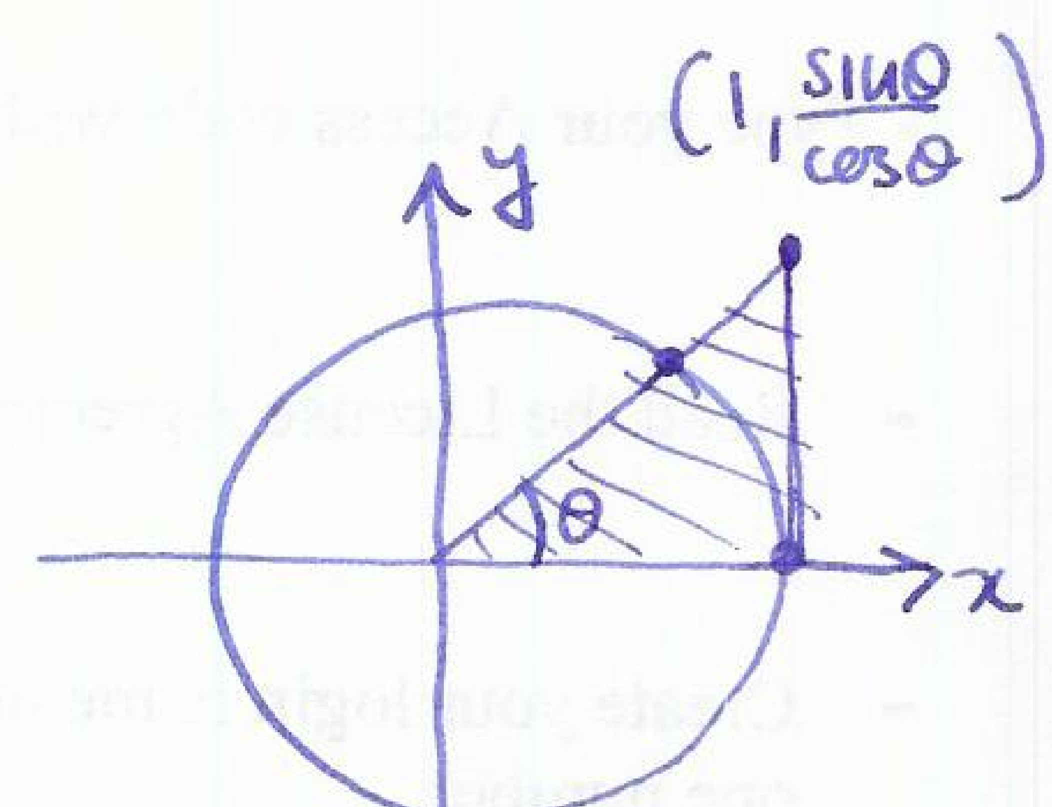
$$\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0 \quad \text{so} \quad \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

Thm $\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Proof (of $\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$) assume $0 < \theta < \frac{\pi}{2}$, consider the following three areas







area of triangle $\le$ area of sector $\le$ area of triangle

$= \frac{1}{2}bh \qquad = \cancel{\frac{2\pi r^2}{\theta}}\ \pi r^2 \cdot \frac{\theta}{2\pi} \qquad = \frac{1}{2}bh$

$$\underbrace{= \frac{1}{2}\cdot 1 \cdot \sin\theta \le \frac{\theta}{2}} \le \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\sin\theta \le \theta \qquad\qquad \theta \le \frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \le 1 \qquad\qquad \cos\theta \le \frac{\sin\theta}{\theta}.$$

$$\Rightarrow \quad \cos\theta \le \frac{\sin\theta}{\theta} \le 1$$