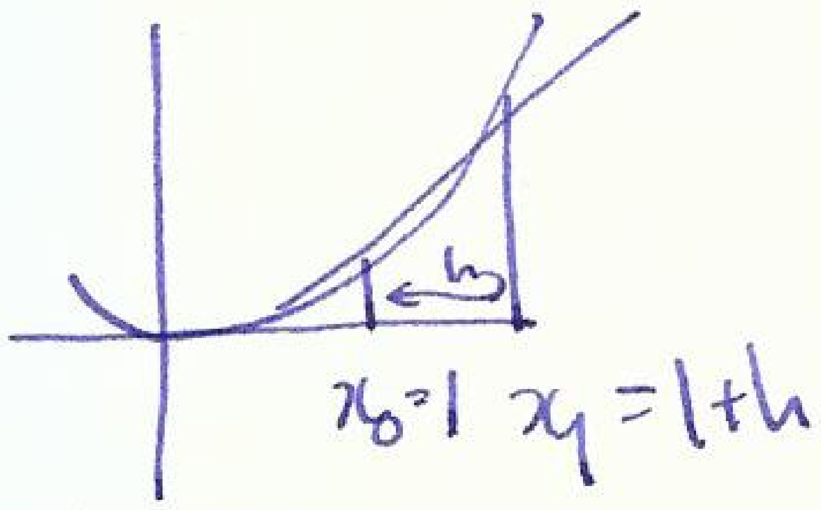
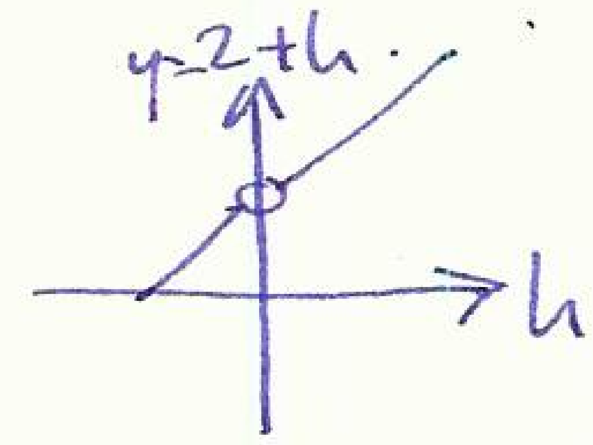
(3) seems to work algebraically



$$x_1 = x_0 + h = 1+h.$$

$$\frac{f(1+h)-f(1)}{1+h-1} = \frac{(1+h)^2-(1)^2}{h}$$

$$= \frac{1+2h+h^2-1}{h} \underset{(h\neq 0)}{=} 2+h$$



__Def__ Let f be a function defined on an interval containing c, but not necessarily defined at c. We say:

"the limit of $f(x)$ as x approaches c is equal to L"

written: $$\lim_{x\to c} f(x) = L$$

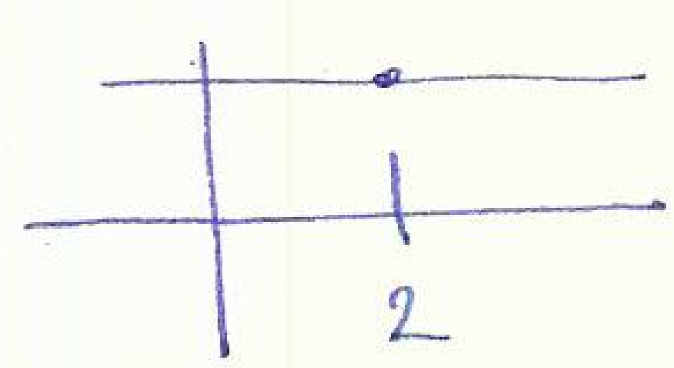
if $|f(x)-L|$ becomes arbitrarily small as x gets close to c.

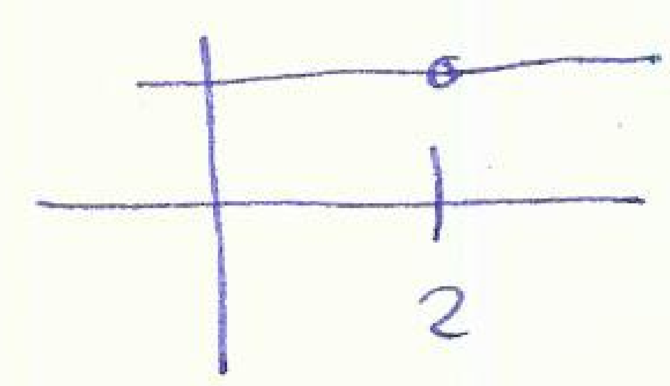
also say "$f(x)$ converges to L as x tends to c"

written: $f(x) \to L$ as $x \to c$.

__Examples__ a) $f(x)=5$, $c=2$

b) $f(x)=\frac{5x}{x}$, $c=2$

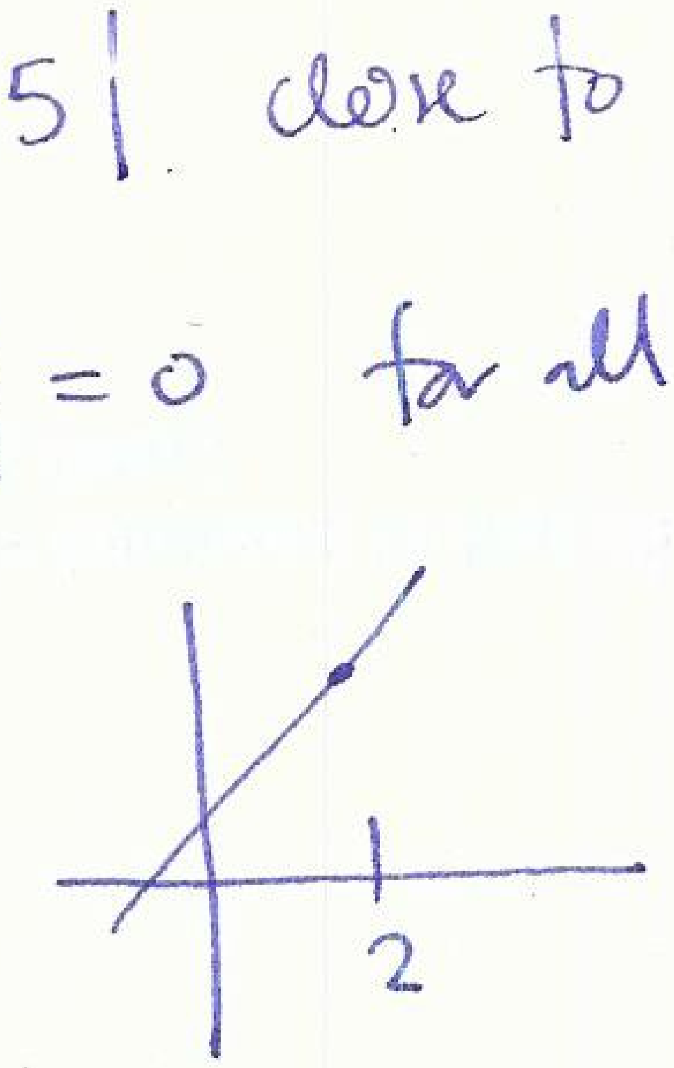




want to show: $|f(x)-5|$ close to 0 if x close to 2.

$|f(x)-5| = |5-5| = 0$ for all x, so this is true.

c) $\lim_{x\to 2} 2x+1 = 5$



want to show $|f(x)-5|$ close to zero ~~as~~ when x close to 2

$$|f(x)-5| = |2x+1-5| = |2x-4| = 2|x-2|$$

x close to 2 $\iff$ $|x-2|$ small.

<u>Useful fact</u>:

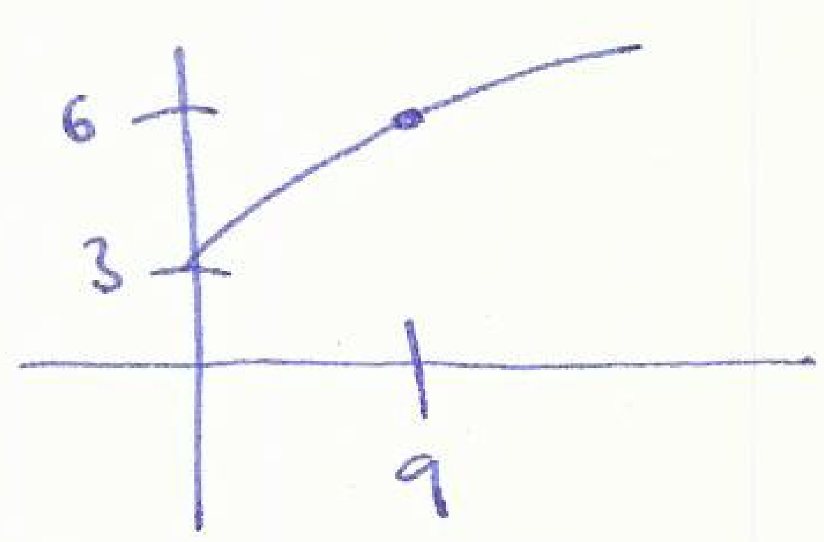
<u>Thm</u> for any constants k, c $\quad \lim_{x \to c} k = k$

$$\lim_{x \to c} x = c$$

Investigating limits: try
- drawing a picture
- calculate close values

<u>Example</u> $\displaystyle \lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$ problem: can't plug in $x=9$ (get $\frac{0}{0}$)

draw picture



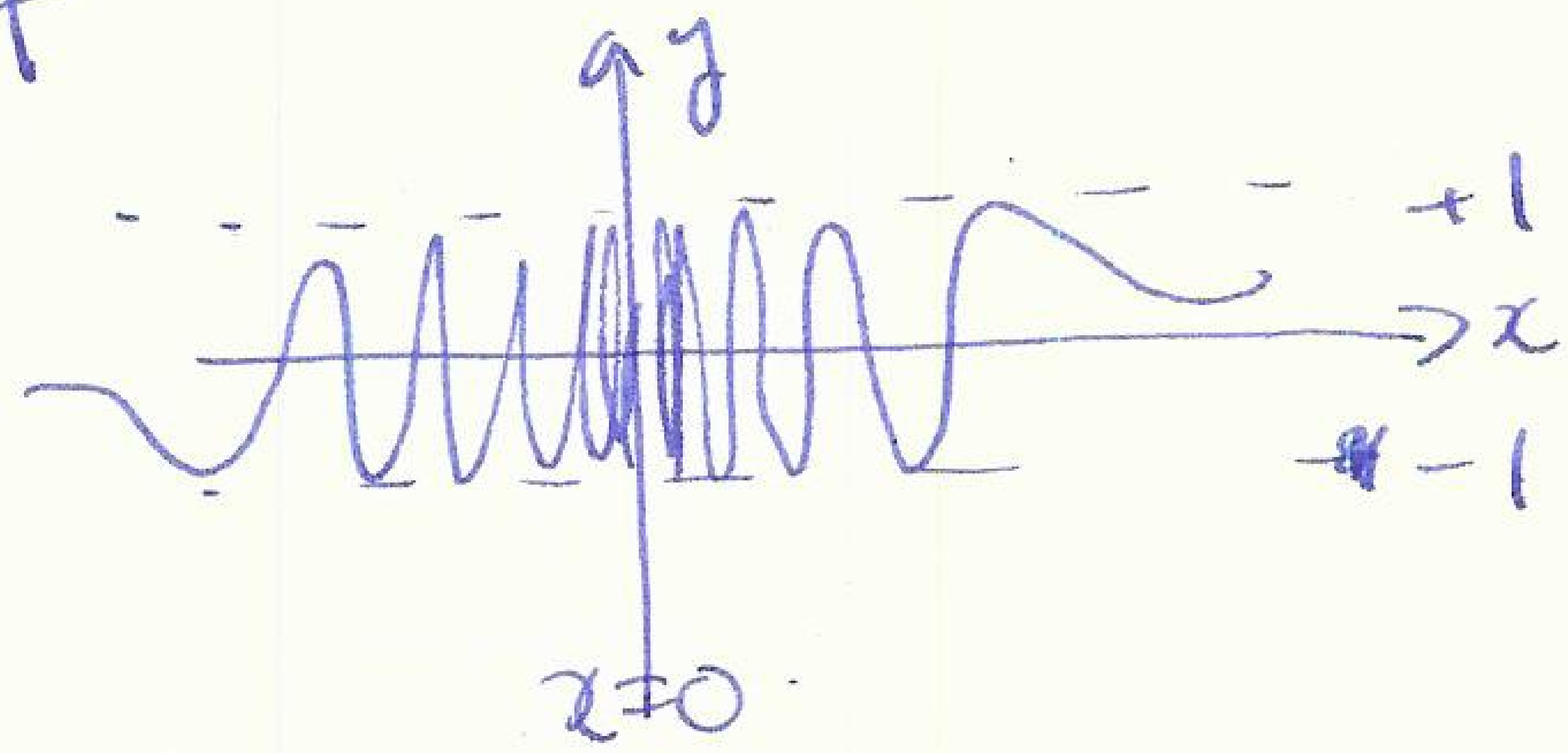
looks like $x = 6$

compute:

x	$\frac{x-9}{\sqrt{x}-3}$
8.9	5.98329
9.1	6.016
8.99	5.99833
9.01	6.00166

Bad example : no limit

$f(x) = \sin(\frac{1}{x})$



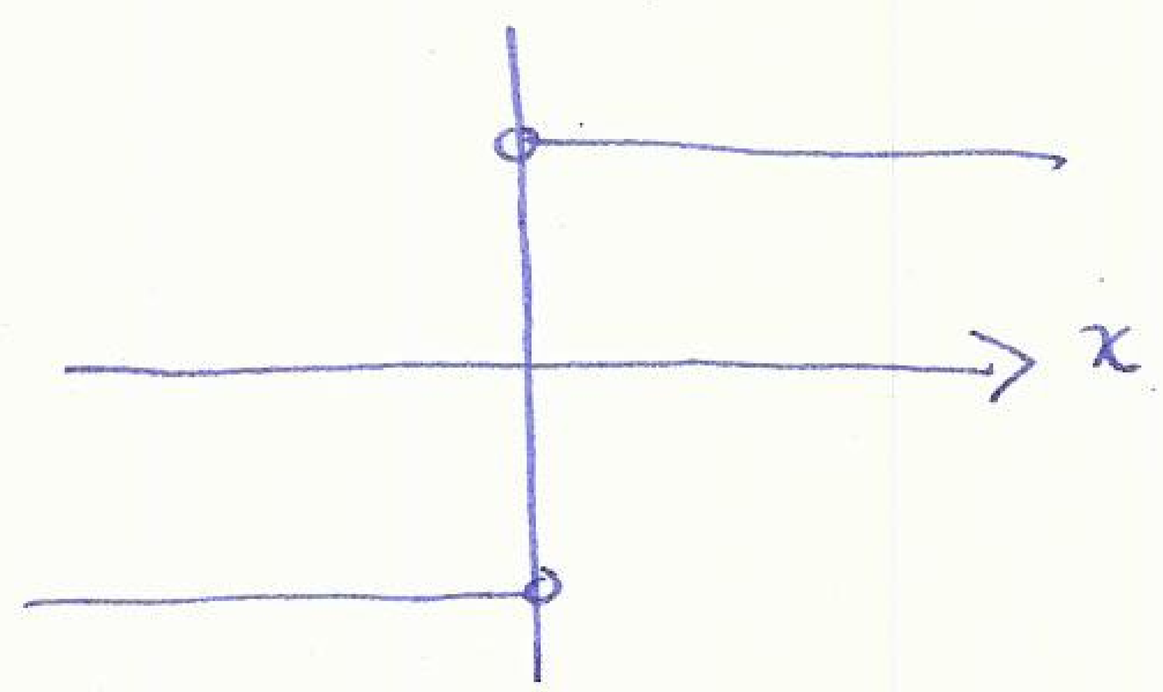
no limit near $x=0$

note : $f(\frac{1}{2\pi n}) = \sin(2\pi n) = 0$

$f(\frac{1}{2\pi n + \frac{\pi}{2}}) = \sin(2\pi n + \frac{\pi}{2}) = 1$

One sided limits

Example :



$f(x) = \frac{x}{|x|}$

$= \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$

sometimes useful to distinguish left from right limits

notation : $\lim_{x \to 0+} f(x)$ means right limit (only consider $x > 0$)

$\lim_{x \to 0-} f(x)$ means left limit (only consider $x < 0$)

note : in order for the limit to exist, the right limit must equal the left limit, i.e. $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x)$.

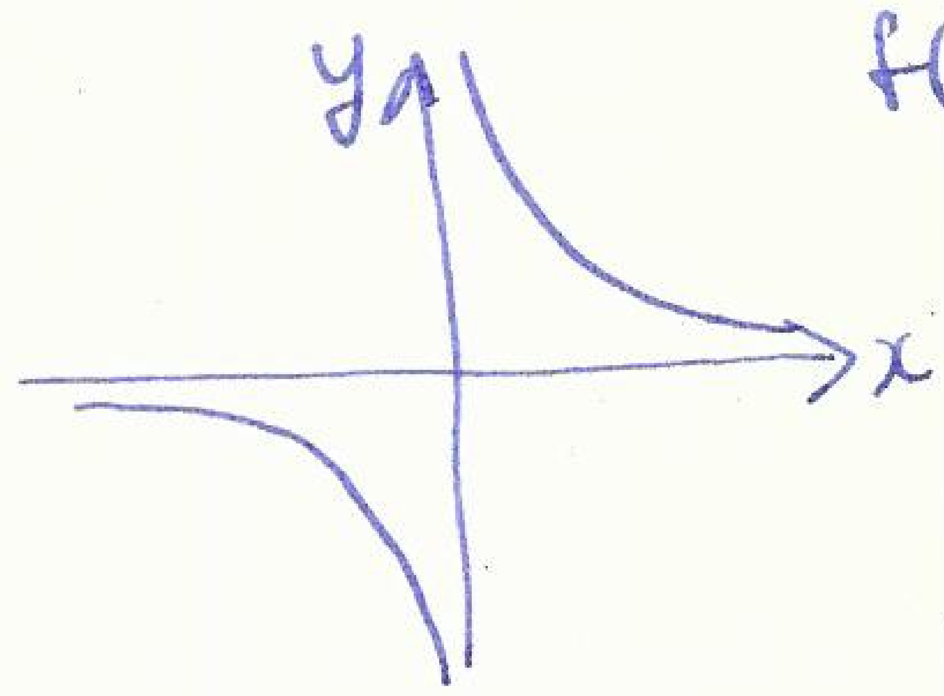
Example $f(x) = \frac{x}{|x|}$. $\left.\begin{array}{l} \lim_{x \to 0+} f(x) = 1 \\ \lim_{x \to 0-} f(x) = -1 \end{array}\right\}$ different so $\lim_{x \to 0} f(x)$ ~~to~~ does not exist DNE

Infinite limits

we say $\lim_{x \to c} f(x) = +\infty$ if $f(x)$ becomes arbitrarily large and positive as $x \to c$.

we say $\lim_{x \to c} f(x) = -\infty$ if $f(x)$ becomes arbitrarily large but negative as $x \to c$.

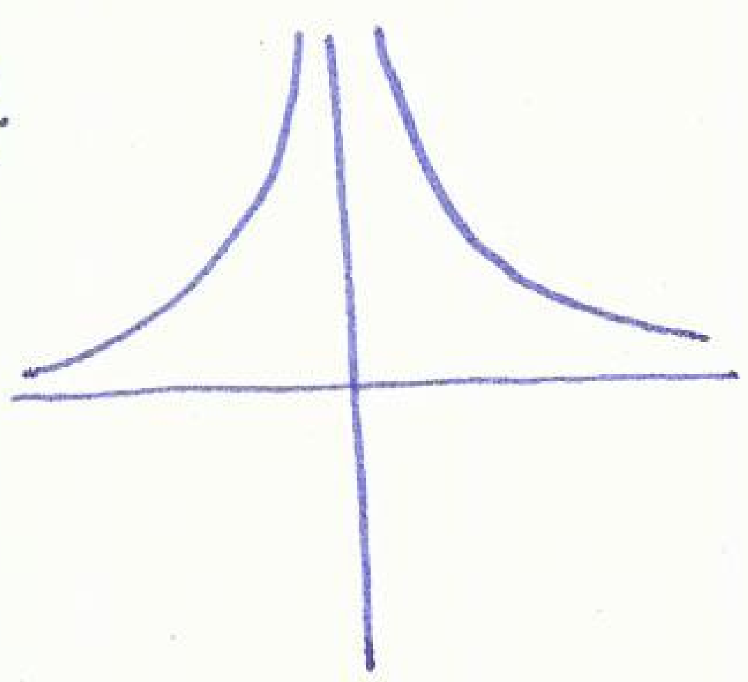
Example



$f(x) = \frac{1}{x}$

$$\left.\begin{aligned} \lim_{x \to 0+} \frac{1}{x} &= +\infty \\ \lim_{x \to 0-} \frac{1}{x} &= -\infty \end{aligned}\right\} \text{ different! so } \lim_{x \to 0} \frac{1}{x} \text{ DNE.}$$

Example

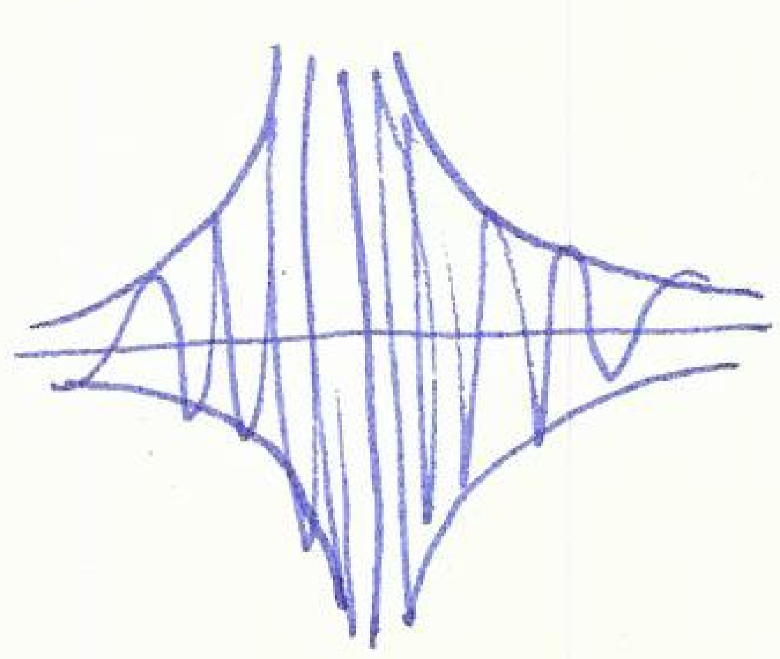


$f(x) = \frac{1}{x^2}$

$$\left.\begin{aligned} \lim_{x \to 0+} \frac{1}{x^2} &= +\infty \\ \lim_{x \to 0-} \frac{1}{x^2} &= -\infty \end{aligned}\right\} \text{ same! so } \lim_{x \to 0} \frac{1}{x^2} = +\infty.$$

Bad example

$f(x) = \frac{1}{x}\sin\left(\frac{1}{x}\right)$



no limit as $x \to 0$.

§2.3 Basic limit laws

Thm assume that $\lim_{x \to c} f(x)$, $\lim_{x \to c} g(x)$ exist (and are finite)

1) sums: $\lim_{x \to c} \left(f(x) + g(x) \right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

2) constant multiple: $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ $\quad$ k constant.

3) products: $\lim_{x\to c} (f(x)\, g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

4) quotients: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ $\quad$ as long as $\lim_{x\to c} g(x) \neq 0$.

Warning: these rules don't work if either of the limits does not exist.

Examples $\quad$ $\lim_{x\to 3} x^2 = \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) = 3.3 = 9.$

$$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\lim_{t\to 2} t+5}{\lim_{t\to 2} 3t} = \frac{7}{6}$$