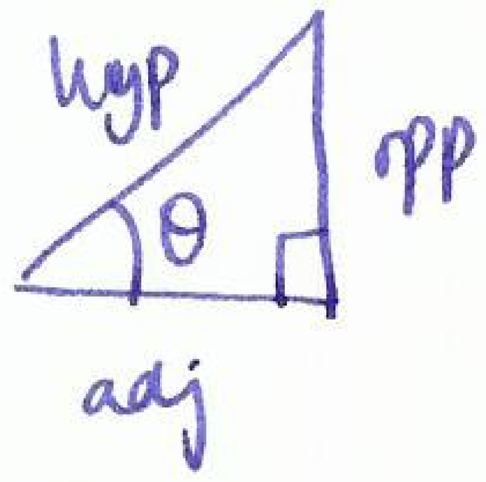
- on unit circle, angle θ is same as angle $\theta + 2\pi$, $\theta + 4\pi$, ... $\theta + 2\pi n$, $n \in \mathbb{Z}$ (integer)

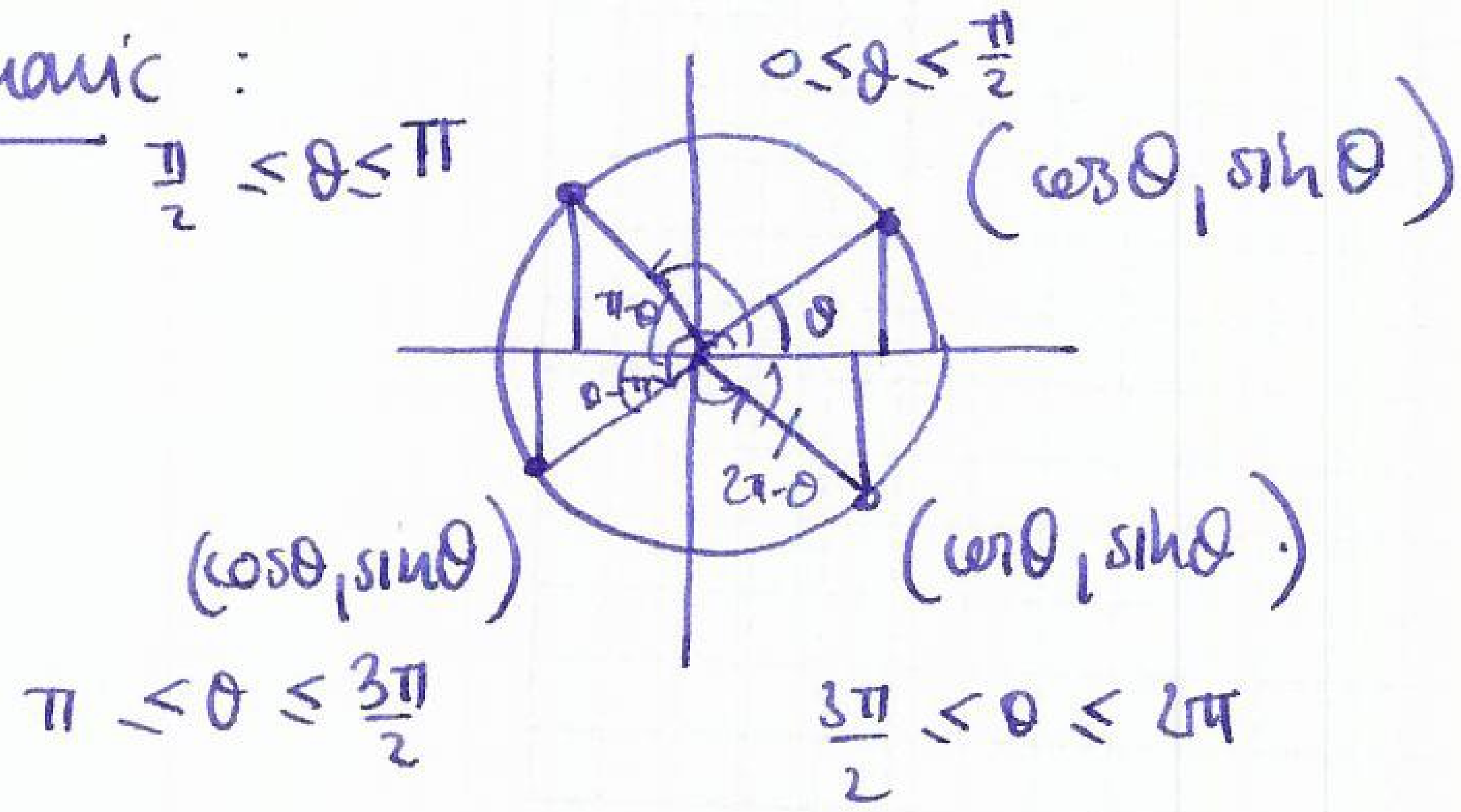
- right angled triangles



$$\sin\theta = \frac{opp}{hyp}, \quad \cos\theta = \frac{adj}{hyp}$$

} can extend these functions to be defined for all values of θ.

mnemonic :



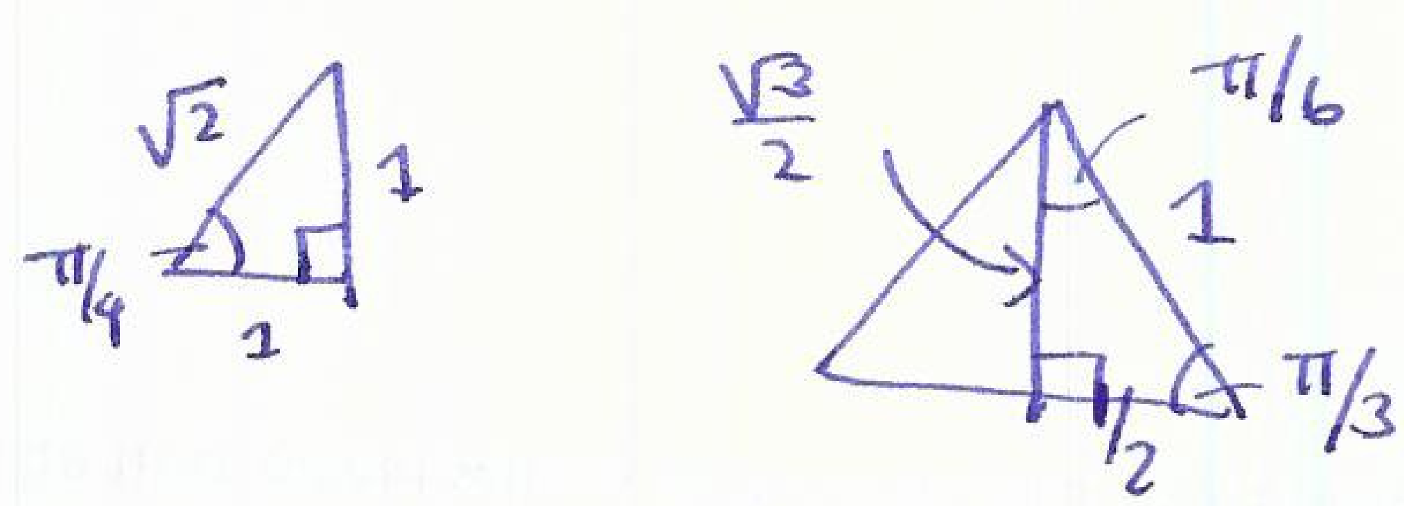
useful facts

- $\sin(-\theta) = -\sin\theta$
  $\cos(-\theta) = \cos\theta$ } for all values of θ.

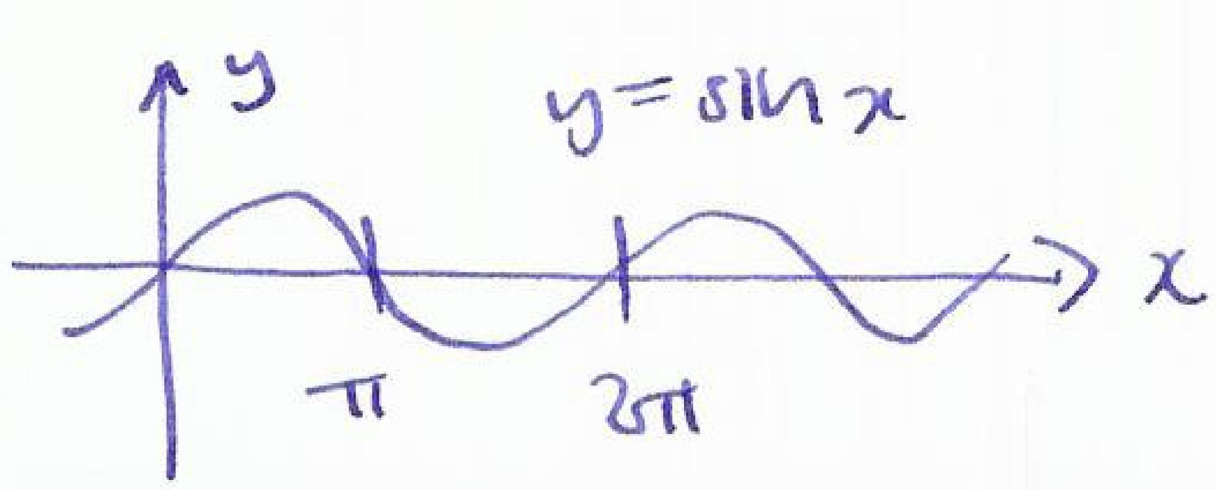
- special values :

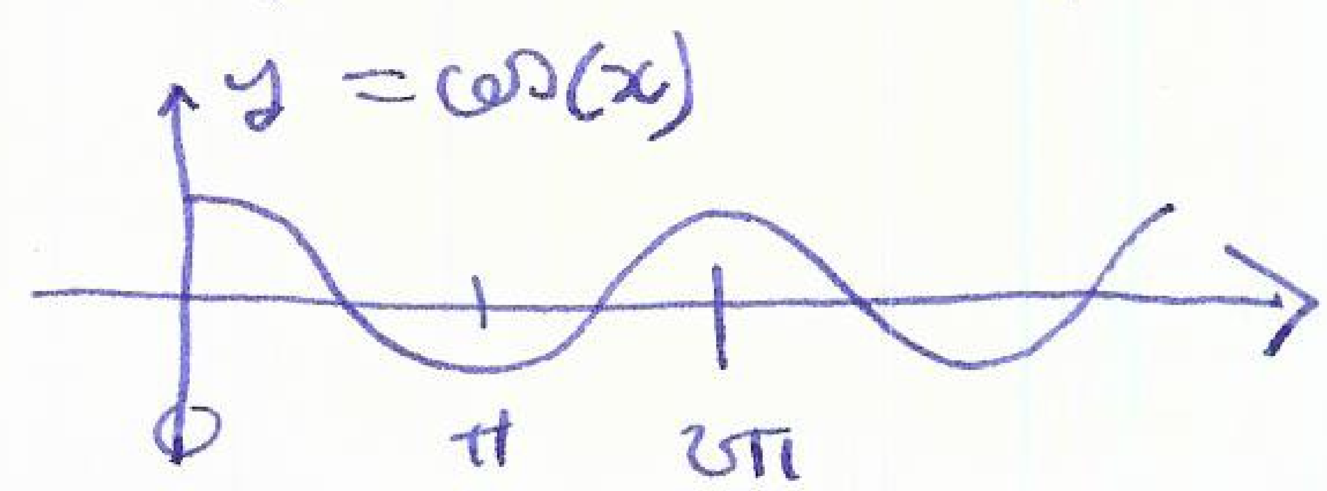
θ	0	$\pi/2$	$\pi/4$	$\pi/3$	$\pi/6$
$\sin\theta$	0	1	$\sqrt{2}/2$	$\sqrt{3}/2$	$1/2$
$\cos\theta$	1	0	$\sqrt{2}/2$	$1/2$	$\sqrt{3}/2$

these come from special triangles :



- the graphs of $\sin x$ and $\cos x$ are periodic with period 2π.





In general a function $f(x)$ is periodic with period T if

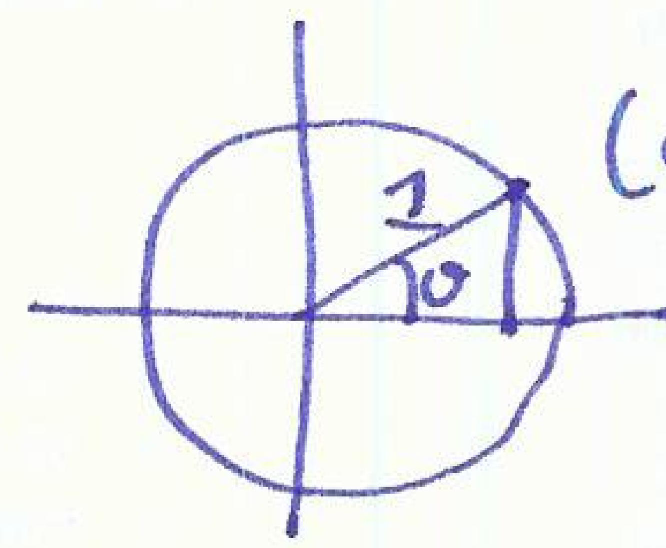
$$f(x+T) = f(x)$$ for all x, and x is the <u>smallest</u> number that works.

<u>other trig functions</u> $\tan(x) = \dfrac{\sin(x)}{\cos(x)} = \dfrac{opp}{adj}$ if $0 < \theta \leq \frac{\pi}{2}$

$\operatorname{cosec} x = \dfrac{1}{\sin x}$ $\quad \sec(x) = \dfrac{1}{\cos(x)}$ $\quad \cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$.

<u>Trig identities</u>

<u>Pythagorean identity</u>: $\cos^2 x + \sin^2 x = 1$



3 identities for the price of one!

$$\frac{\sin^2 x + \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} \leftrightarrow 1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \leftrightarrow \tan^2 x + 1 = \sec^2 x$$

<u>Double angle</u>

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

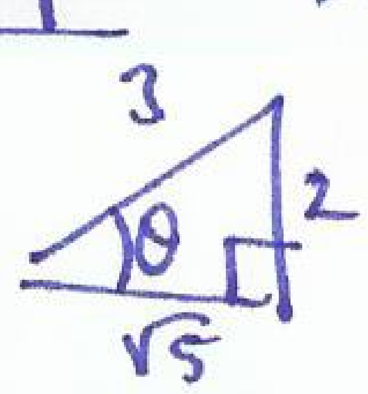
<u>Addition</u>

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

shift: $\sin(x + \frac{\pi}{2}) = \cos(x)$

<u>Example</u> suppose $\sin\theta = \frac{2}{3}$ find $\cos\theta$, $\tan\theta$, $\sin 2\theta$



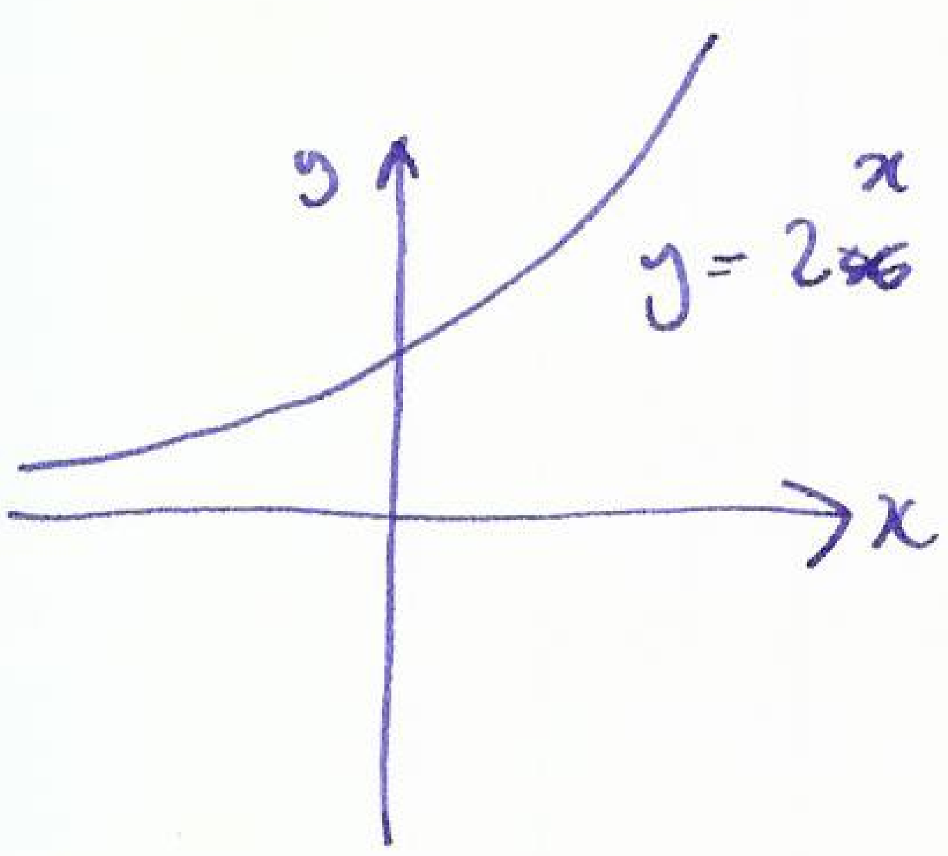
$\cos\theta = \frac{\sqrt{5}}{3}$ $\quad \tan\theta = \frac{2}{\sqrt{5}}$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{2}{3} \cdot \frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}.$$

§1.6 Exponential and logarithmic functions

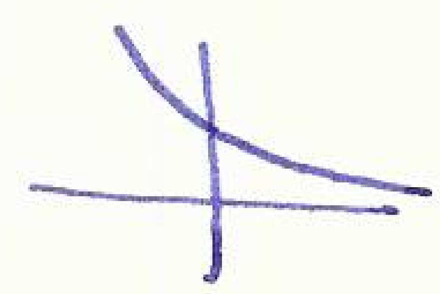
Example $x \mapsto 2^x$

x	-2	1	0	1	2	3
$f(x)$	$1/4$	$1/2$	1	2	4	8



can use any positive number instead of 2

$$f(x) = b^x \qquad (b>0)$$

eg $b = \frac{1}{2}$ 

useful properties

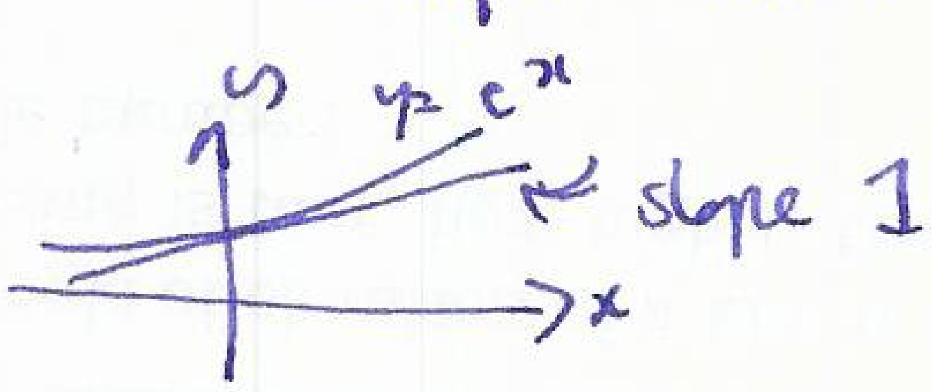
- positive $b^x > 0$ for all x.
- b^x increasing if $b > 1$
  decreasing if $0 < b < 1$
- b^x grows faster than any polynomial x^n.
- exponent rules

$$b^0 = 1, \quad b^x b^y = b^{x+y}, \quad b^{-x} = \frac{1}{b^x}, \quad \frac{b^x}{b^y} = b^{x-y}$$

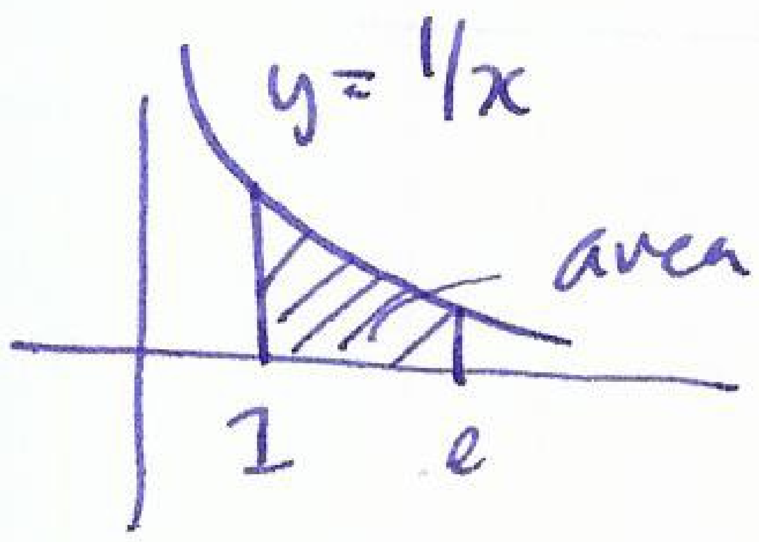
$$(b^x)^y = b^{xy}, \quad b^{1/n} = \sqrt[n]{b}.$$

- there is a special exponential function e^x, $e = 2.71828\ldots$

key properties: ① e is the unique number such that e^x has slope 1 at $x=0$



②



e is the unique number such that the area under $y = 1/x$ between 1 and e has area 1.

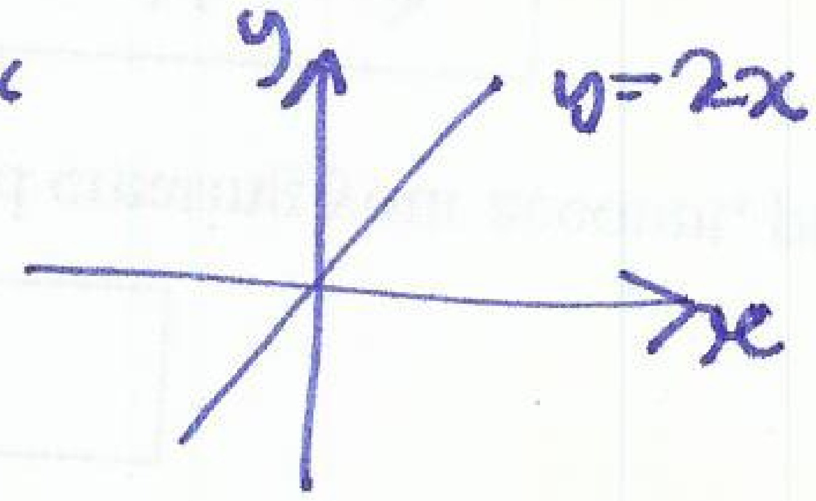
Logarithms

The logarithm is the <u>inverse function</u> for the exponential function.

Recall domain $\underset{f^{-1}}{\overset{f}{\rightleftarrows}}$ range.

Warning $f^{-1}(x) \neq \dfrac{1}{f(x)}$

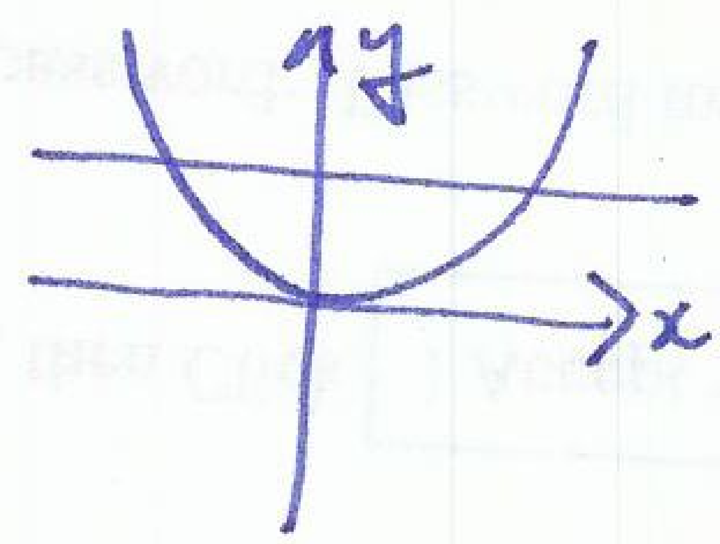
Example $f(x) = 2x$



$f^{-1}(x) = \frac{1}{2}x$

$3 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 6$

Bad example $f(x) = x^2$



$1 \overset{f}{\longmapsto} 1$

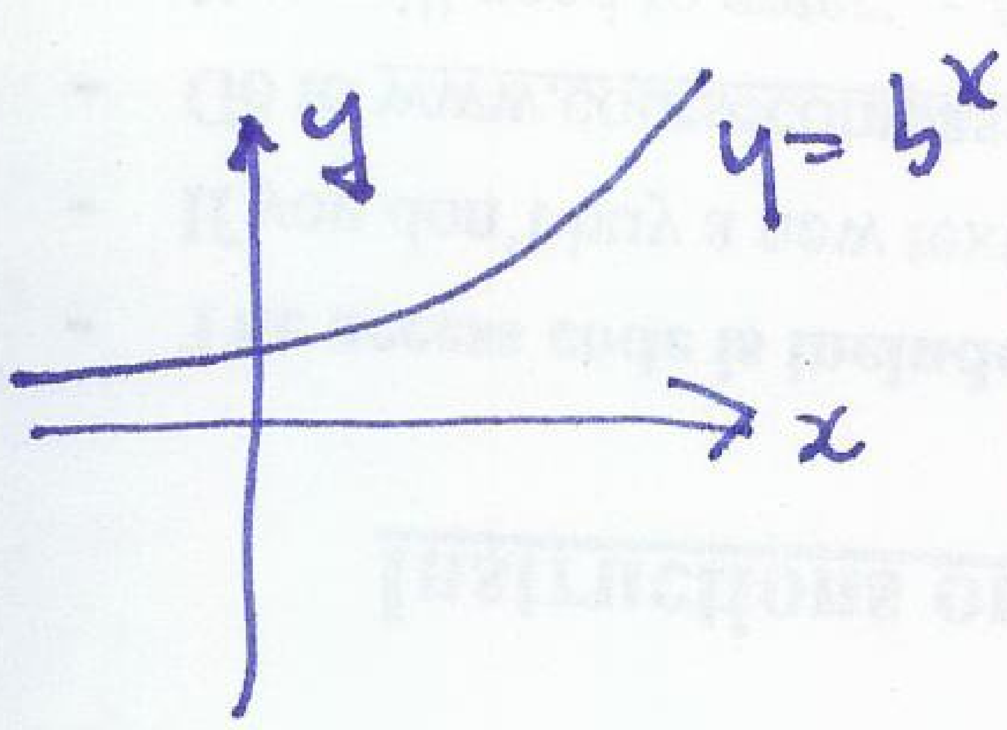
$-1 \longmapsto 1$

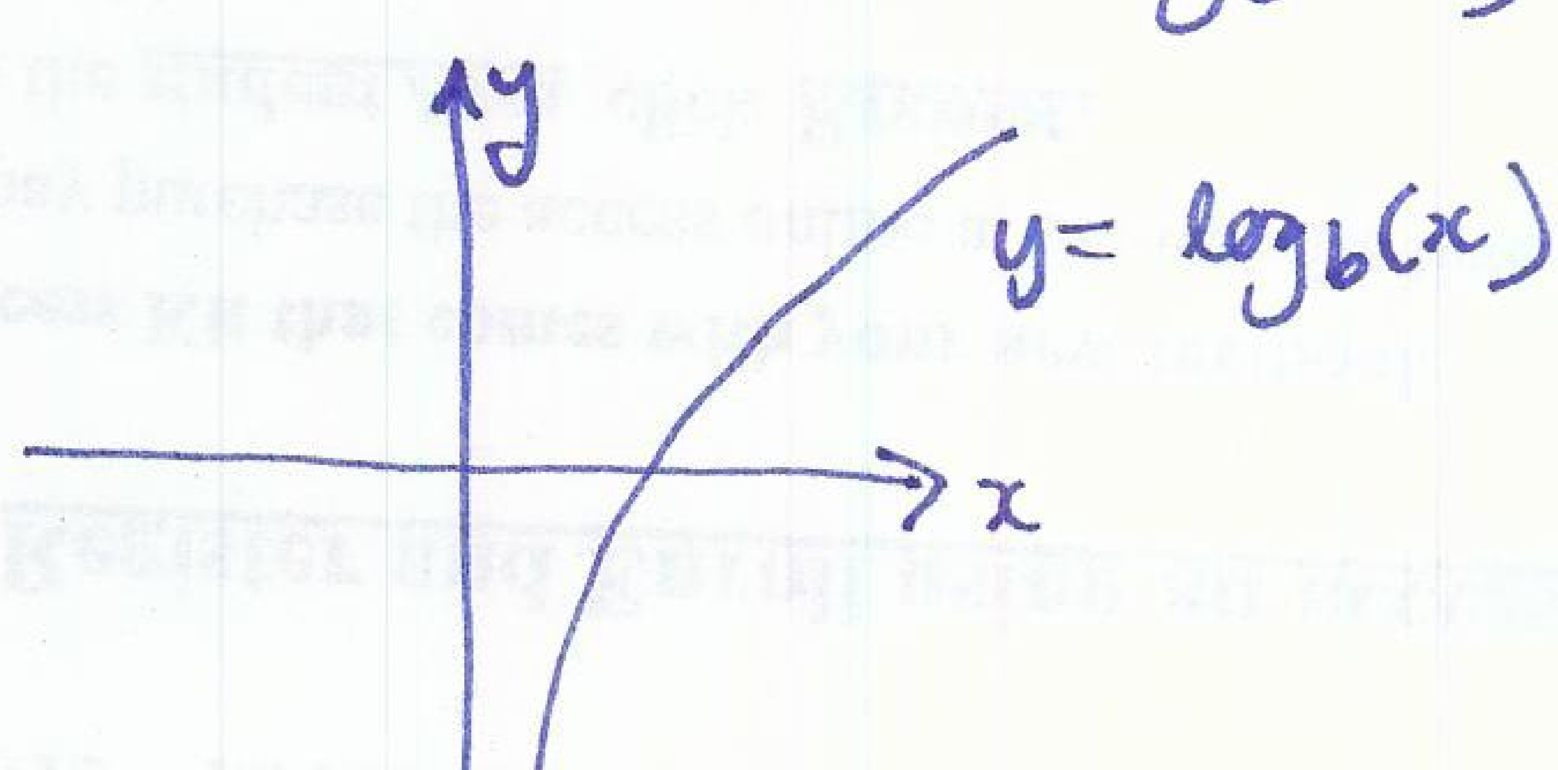
$? \xleftarrow{f^{-1}}$

A function has an inverse iff it passes the <u>horizontal line test</u>

i.e. every horizontal line hits the graph of the function exactly once.

The exponential function $y = b^x$ has an inverse called the logarithm (to base b) written $f^{-1}(x) = \log_b(x)$





Useful fact: graph of $f^{-1}(x)$ is graph of $f(x)$ reflected through $y = x$.

special logarithm with $b=e$ is called <u>natural logarithm</u>

notation: $\log_e(x) = \ln(x)$

- inverse function properties: $f(f^{-1}(x)) = x$

$$f^{-1}(f(x)) = x$$

So $b^{\log_b(x)} = x$ and $\log_b(b^x) = x$

<u>logarithm rules:</u>

$\cancel{\log_b(1)=0}$ $\quad b^0 = 1 \leftrightarrow \log_b(1) = 0$

$$b^1 = b \leftrightarrow \log_b(b) = 1$$

$$b^x b^y = b^{x+y} \leftrightarrow \log_b(st) = \log_b(s) + \log_b(t)$$

$$\frac{1}{b^x} = b^{-x} \leftrightarrow \log_b\left(\frac{1}{t}\right) = -\log_b(t)$$

$$\frac{b^x}{b^y} = b^{x-y} \leftrightarrow \log\left(\frac{s}{t}\right) = \log(s) - \log(t)$$

$$(b^x)^y = b^{xy} \leftrightarrow \log_b(s^t) = t\log_b(s)$$

converting between bases:

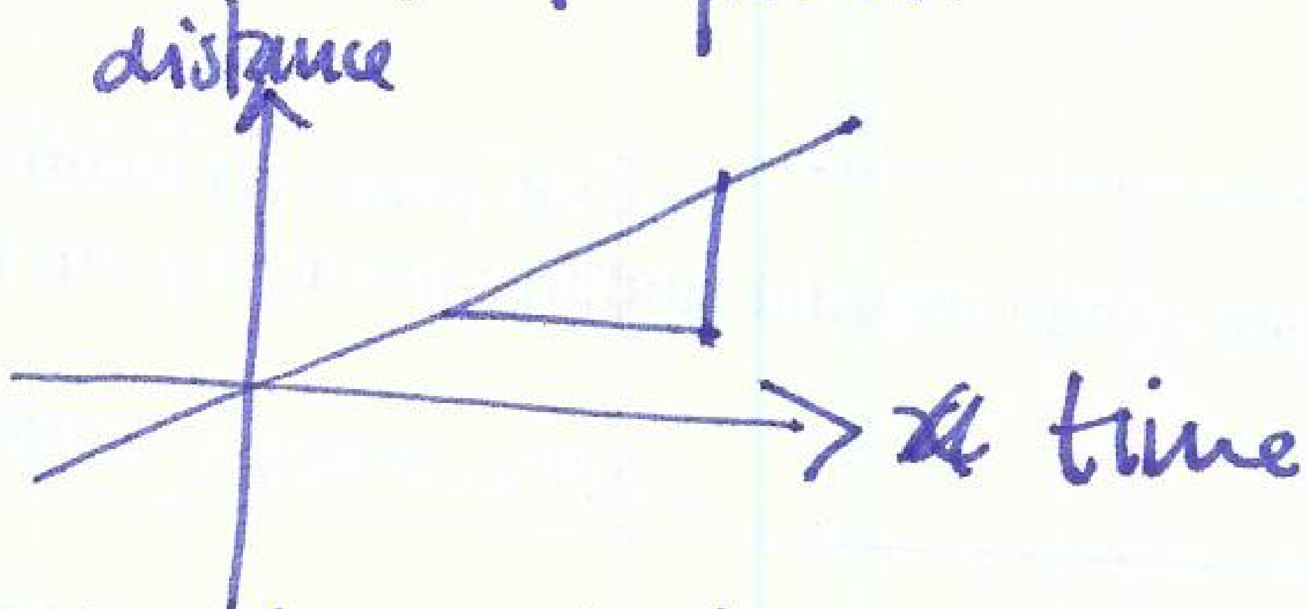
$$\log_b x = \frac{\log_a x}{\log_a b} \text{ for any } a.$$

so $\log_b x = \dfrac{\ln(x)}{\ln(b)}$

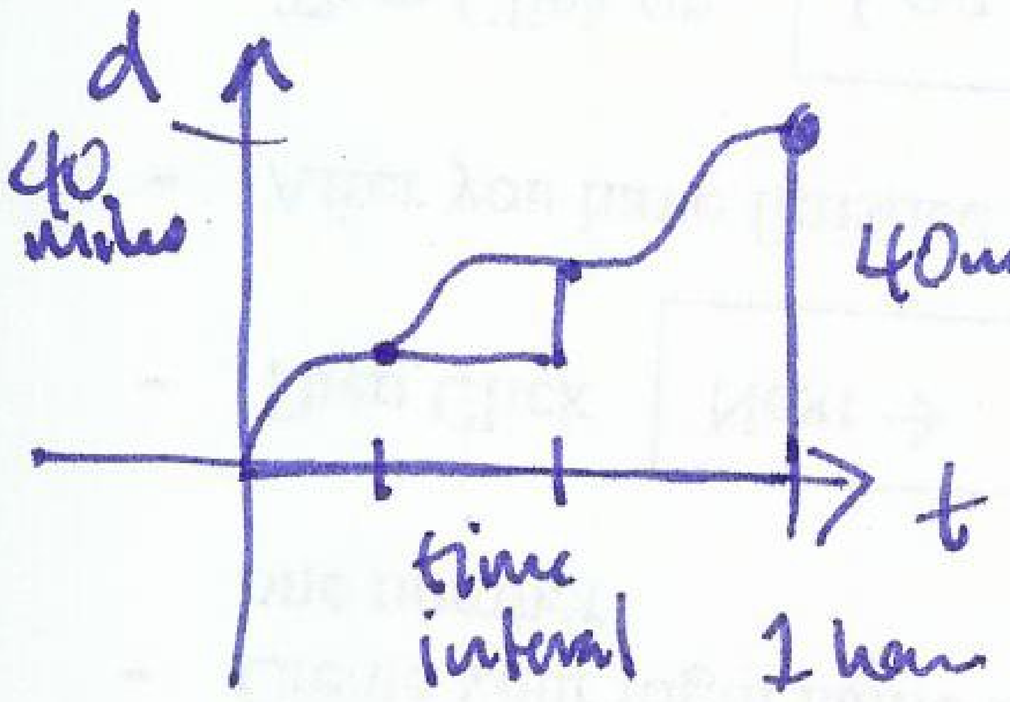
§2.1 Limits, rates of change, tangent lines

<u>motivation</u> : <u>velocity</u> example : driving at constant speed.

velocity $= \dfrac{\text{distance}}{\text{time}} =$ slope of line

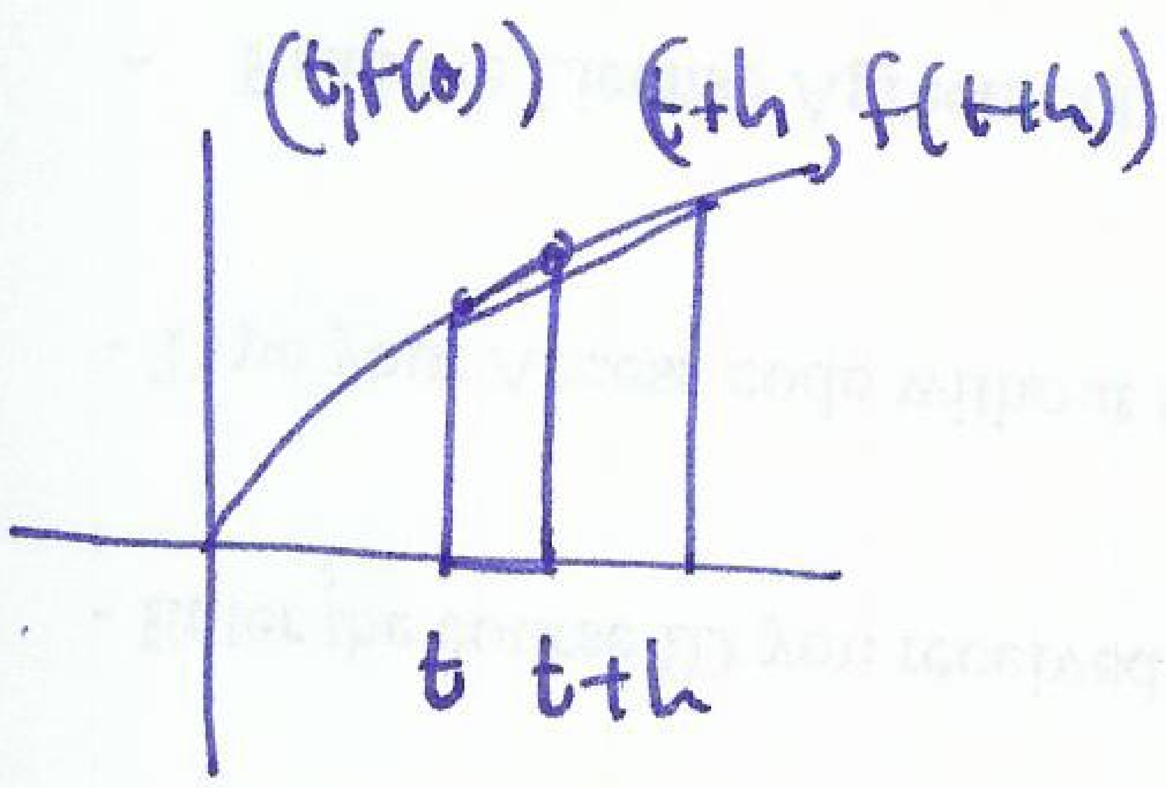


<u>problem</u> : what happens if you don't travel at constant speed?



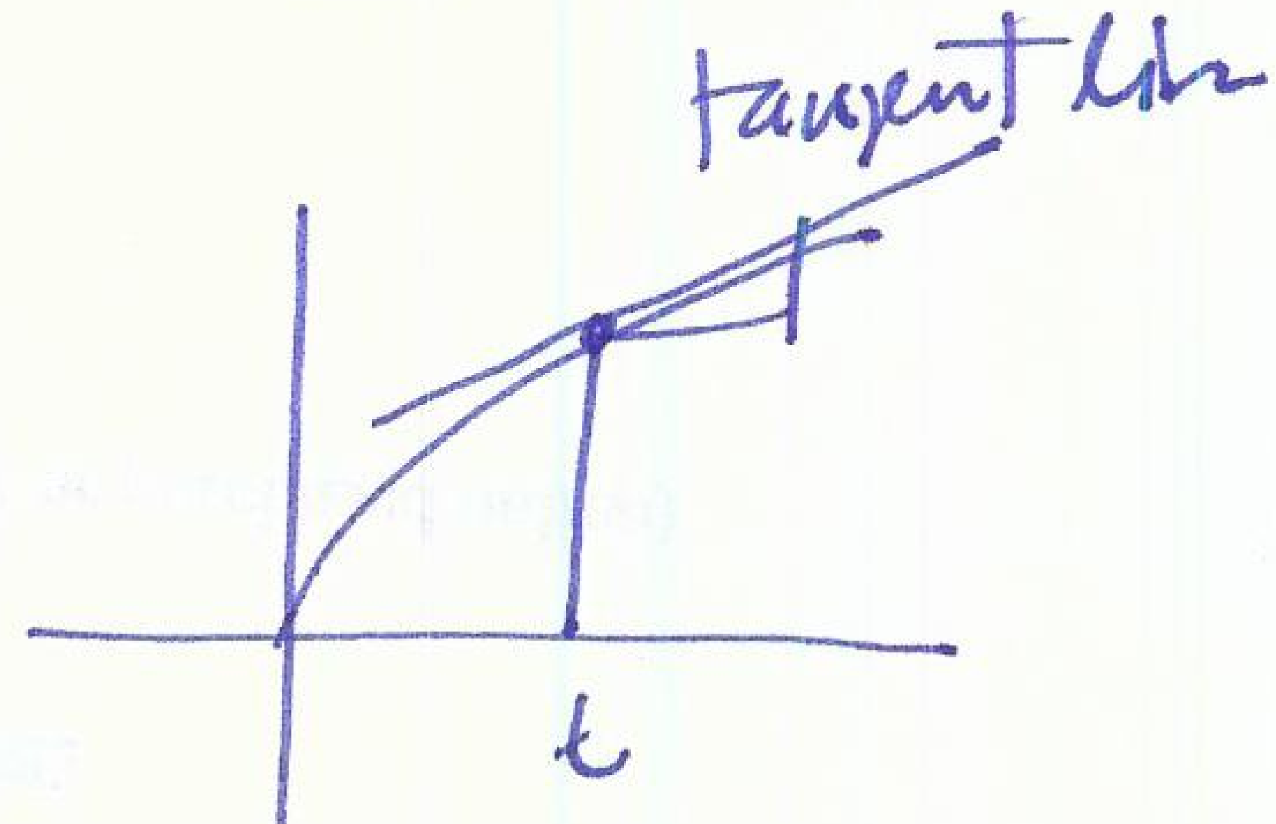
average speed $= \dfrac{\text{distance travelled}}{\text{length of time interval}}$.

we can look at average speed over <u>any</u> time interval, including very short ones.



average speed on time interval $[t, t+h]$ is $\dfrac{\Delta d}{\Delta t} = \dfrac{f(t+h) - f(t)}{t+h-t} = \dfrac{f(t+h) - f(t)}{h}$

<u>Q</u>: what is the speed at time t?
(sometimes called instantaneous speed)



<u>A</u> given by slope of tangent line at t

<u>idea: / hope</u>: as length of interval $[t, t+h]$ gets small the average speed gets close to slope of tangent line.

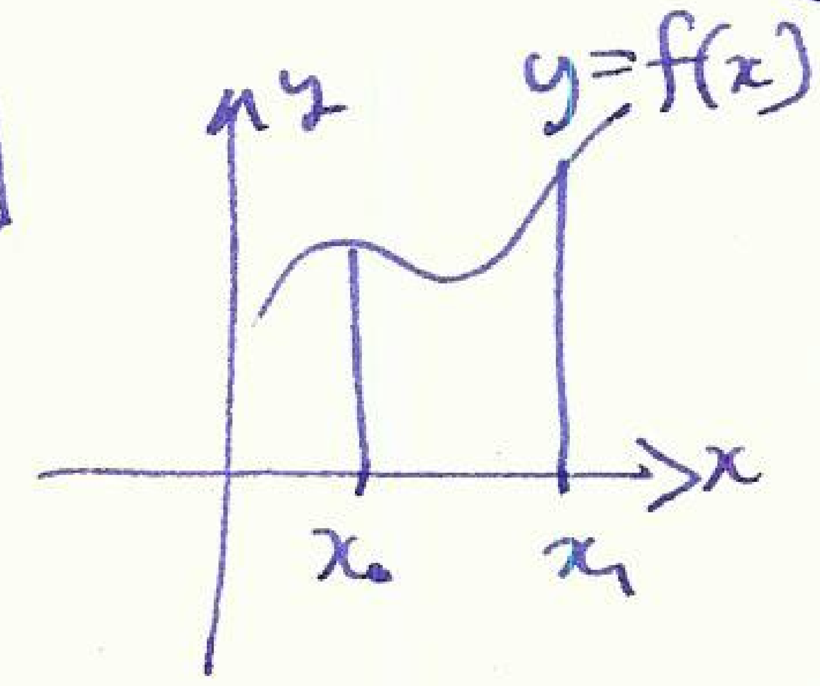
This works for "nice" functions.

<u>observation</u> : this works for any function, not just speed.

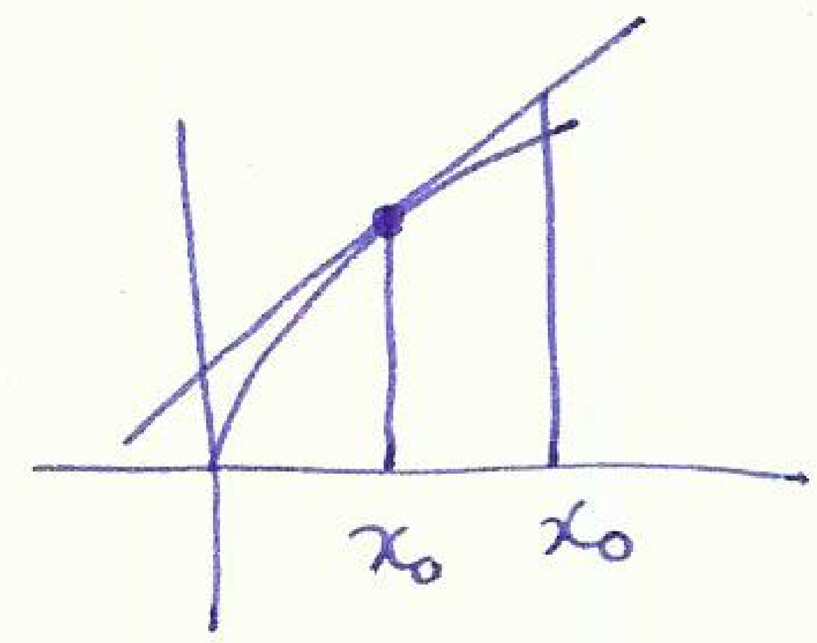
<u>Summary</u>

average rate of change of $f(x)$ over interval $[x_0, x_1]$

is $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$



§2.2 Limits



<u>aim</u>: want to find slope of tangent lines:

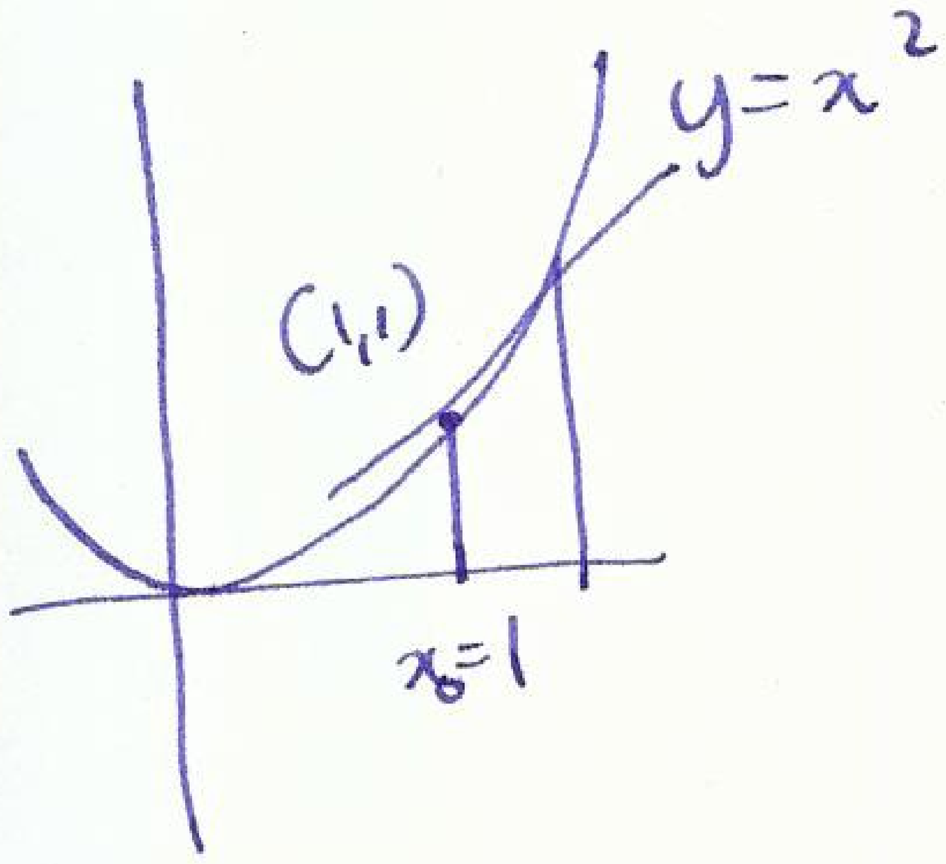
<u>know</u>: average rate of change: $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

<u>Q</u>: why can't we just set $x_1 = x_0$? <u>A</u>: No. $\dfrac{f(x_0)-f(x_0)}{x_0-x_0} = \dfrac{0}{0}$ undefined.

<u>Observations</u>

① if we draw pictures, the average slope gets closer to the slope of the tangent line as the length of the interval gets small.

② seems to work for sample calculations too.



$x_1 = 2$: $\dfrac{f(2)-f(1)}{2-1} = \dfrac{4-1}{1} = 3.$

$x_1 = \frac{3}{2}$: $\dfrac{f(\frac{3}{2})-f(1)}{\frac{3}{2}-1} = \dfrac{\frac{9}{4}-1}{\frac{1}{2}} = \dfrac{5}{2} \simeq 2.5$

$x_1 = 1.1$ $\dfrac{1.21-1}{0.1} = 2.1$

$x_1 = 1.01$ $\dfrac{1.0201-1}{0.01} \approx 2.$