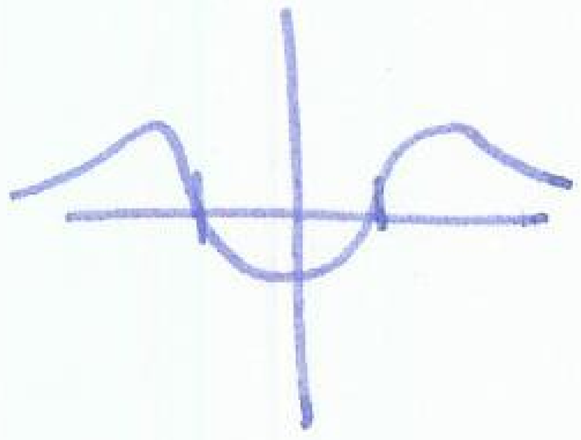
Midterm 3 Solutions

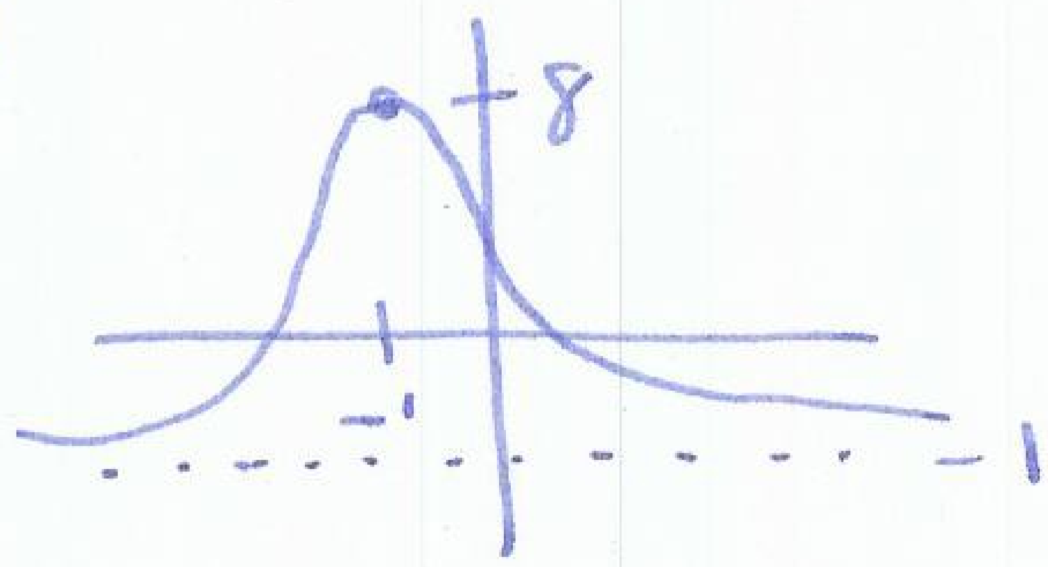
Q1 a) $[-1,1]$ b) $[-10,-1]$, $[1,10]$ c) 0 d) 0

e)



f) $-2, 0, 2$

Q2

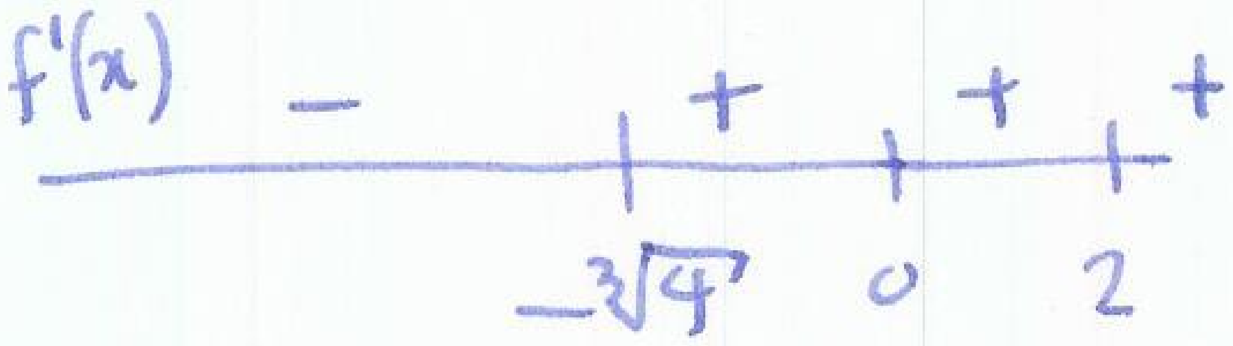


Q3 a) horizontal asymptotes: $\lim_{x\to+\infty} \frac{x}{8-x^3} = 0$ $\lim_{x\to-\infty} \frac{x}{8-x^3} = 0$

vertical asymptotes: $8-x^3=0 \Rightarrow x=2$.

b) $f'(x) = \dfrac{(8-x^3)\cdot 1 - x(-3x^2)}{(8-x^3)^2} = \dfrac{8+2x^3}{(8-x^3)^2} \Rightarrow x = -\sqrt[3]{4}$

c) $f'(x)$

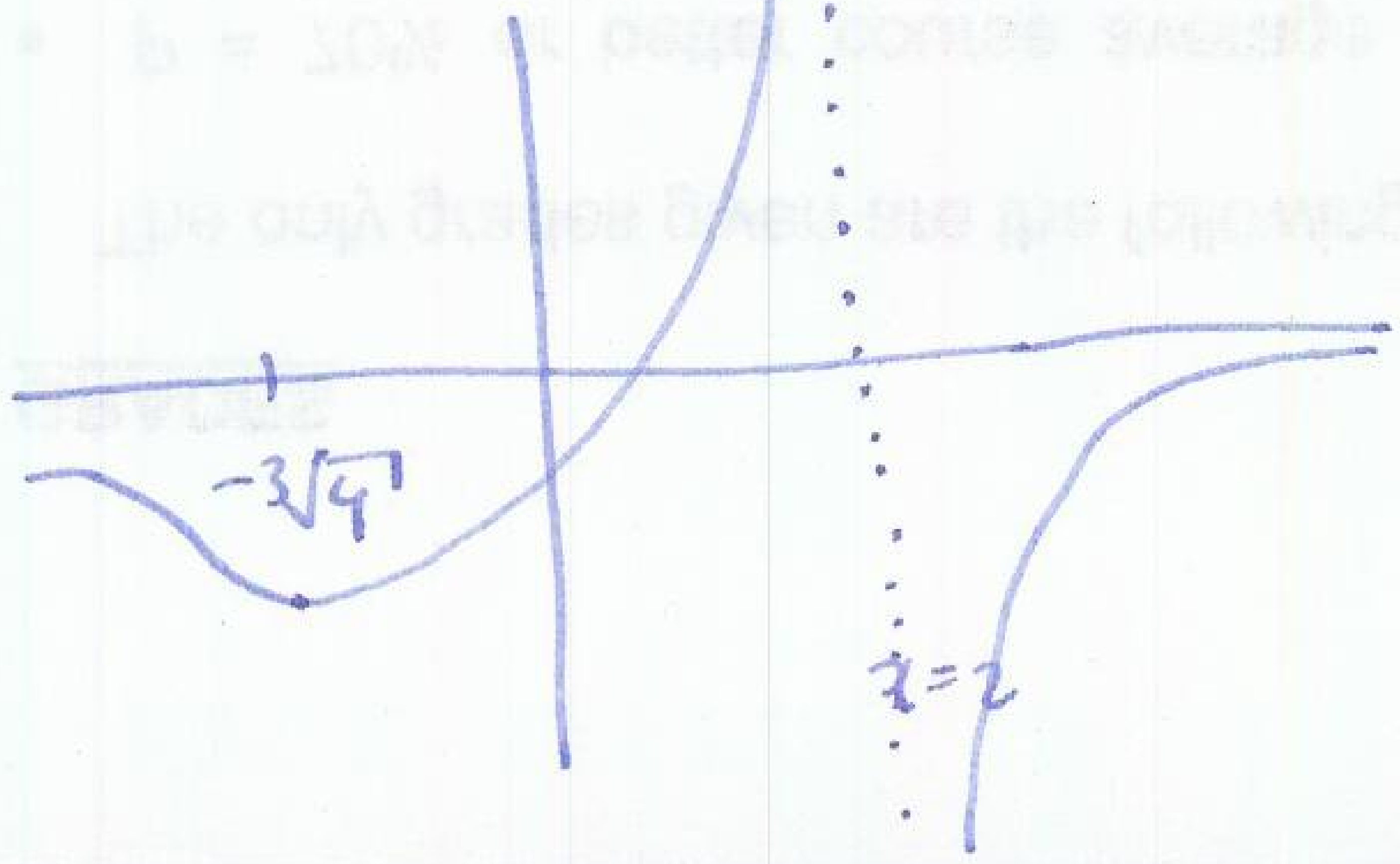


d) $f''(x) = \dfrac{(8-x^3)^2(6x^2) - 2x^3 \cdot 2(8-x^3)(-3x^2)}{(8-x^3)^4}$

$f''(-\sqrt[3]{4}) = \dfrac{12^2 \cdot 6 \cdot \sqrt[3]{16} + 2\cdot(-4)\cdot 12\cdot 3\cdot\sqrt[3]{16}}{+} = \dfrac{12\cdot 6\cdot\sqrt[3]{16}(12-4)}{+} > 0$

$\Rightarrow -\sqrt[3]{4}$ local min.

e)



Q4 a) $g(x) = (x^2 - 2x)\, e^{-2x}$

$$g'(x) = (2x-2)e^{-2x} + (x^2-2x)(-2e^{-2x}) = 2e^{-2x}(x - 1 - x^2 + 2x)$$

$$= -2e^{-2x}(x^2 \cancel{4} - 3x + 1)$$

critical points: solve $g'(x) = 0$: $x = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

sign of $g'(x)$: $-$ | $+$ | $-$ with marks at $\frac{3-\sqrt5}{2}$, $\frac{3+\sqrt5}{2}$ $\Rightarrow$ $\frac{3-\sqrt5}{2}$ local min

$\frac{3+\sqrt5}{2}$ local max

b) g increasing on $\left(\frac{3-\sqrt5}{2}, \frac{3+\sqrt5}{2}\right)$ and decreasing on $\left(-\infty, \frac{3-\sqrt5}{2}\right) \cup \left(\frac{3+\sqrt5}{2}, \infty\right)$

c) $g''(x) = 4e^{-2x}(x^2 - 3x + 1) - 2e^{-2x}(2x - 3)$.

$$= 2e^{-2x}(2x^2 - 6x + 2 - 2x + 3) = 2e^{-2x}(2x^2 - 8x + 5)$$

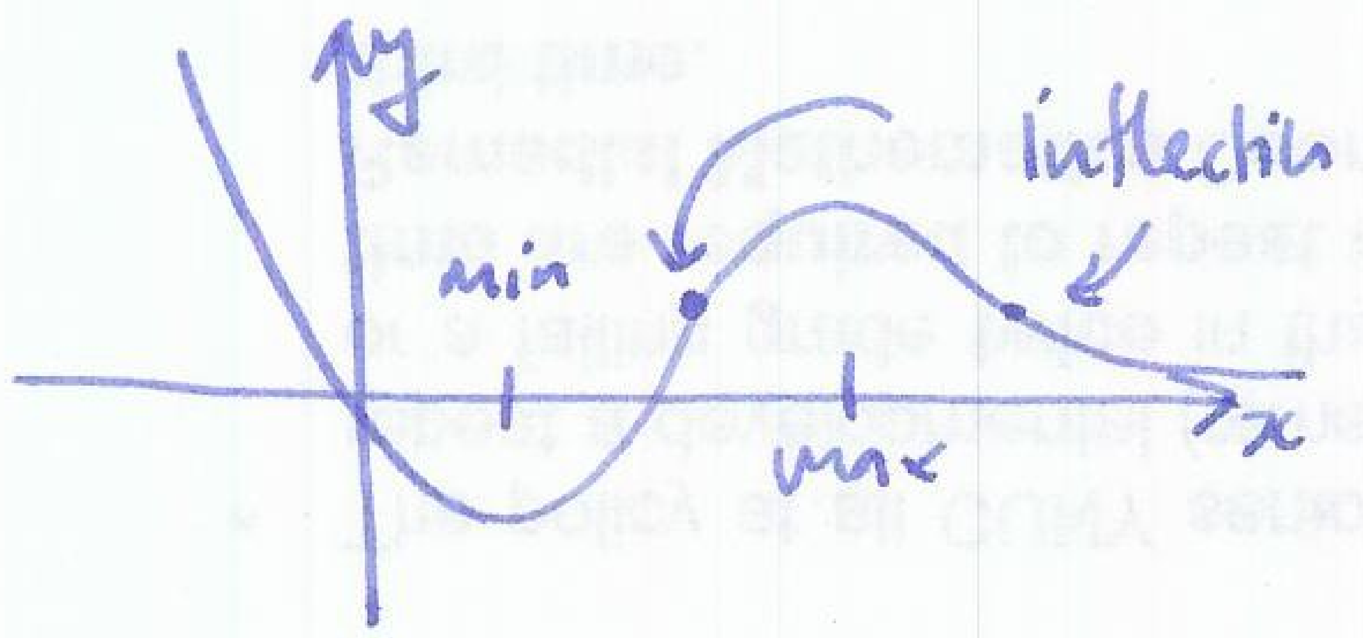
points of inflection $g''(x) = 0$: $x = \dfrac{8 \pm \sqrt{64 - 40}}{2} = 4 \pm \sqrt{6}$.

d) sign of $g''(x)$: $+$ | $-$ | $+$ with marks at $4-\sqrt6$, $4+\sqrt6$

concave up: $(-\infty, 4-\sqrt6) \cup (4+\sqrt6, \infty)$

concave down: $(4-\sqrt6, 4+\sqrt6)$

e)



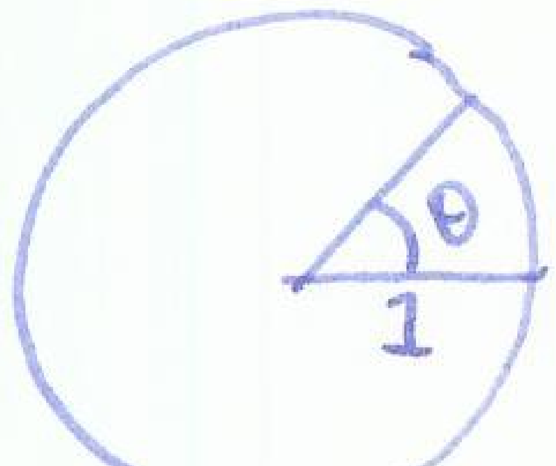
Q5 $f'(x) > 0 \Rightarrow$ increasing $\Rightarrow$ max at 4

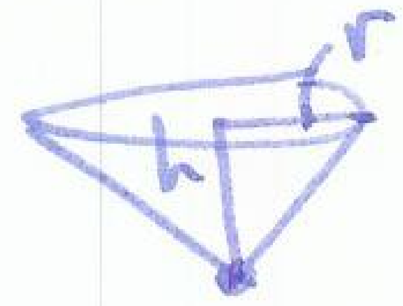
Q6

Q5 $f'(x)$ always +ve $\Rightarrow$ function increasing

$\Rightarrow$ max value at 4.

Q6





volume of cone $= \frac{1}{3}\pi r^2 h$

circumference of cone $= 2\pi - \theta = 2\pi r \Rightarrow r = \frac{2\pi - \theta}{2\pi}$

$r^2 + h^2 = 1 \Rightarrow h = \sqrt{1-r^2}$.

$$V = \frac{1}{3}\pi\left(\frac{2\pi-\theta}{2\pi}\right)^2\left(1-\left(\frac{2\pi-\theta}{2\pi}\right)^2\right)^{1/2} = \frac{1}{3}\pi r^2\sqrt{1-r^2} = \frac{1}{3}\pi\sqrt{r^4-r^6}$$

$$\frac{dV}{d\theta} = \frac{dV}{dr}\cdot\frac{dr}{d\theta} = \frac{1}{3}\pi\frac{1}{2}\left(r^4-r^6\right)^{-1/2}\left(4r^3-6r^5\right)\cdot -\frac{1}{2\pi}$$

$$= \frac{1}{12}\,\frac{4r^3-6r^5}{\sqrt{r^4-r^6}}$$

solve $\frac{dV}{d\theta} = 0 \Rightarrow 4r^3 = 6r^5 \Rightarrow r^2 = \frac{4}{6} = \frac{2}{3}$ $\quad r = \sqrt{\frac{2}{3}}$

$\Rightarrow \frac{2\pi-\theta}{2\pi} = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \theta = 2\pi\left(1 - \sqrt{2}/\sqrt{3}\right)$.

Q7 a) $$\lim_{x\to\pm\infty}\frac{12x+1}{\sqrt{4x^2+4}} = \lim_{x\to-\infty}\frac{12+1/x}{\sqrt{4+4/x^2}} = 6.$$

b) $$\lim_{x\to0+}\sqrt{x}\ln(x) = \lim_{x\to0+}\frac{\ln(x)}{x^{-1/2}} \underset{\text{l'Hôpital}}{=} \lim_{x\to0+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to0+} -2x^{1/2} = 0$$

c) $$\lim_{x\to0}\frac{xe^x - e^x + 1}{x(e^x-1)} \underset{\text{l'Hôpital}}{=} \lim_{x\to0}\frac{xe^x+e^x-e^x}{xe^x+e^x-1} = \lim_{x\to0}\frac{1}{1+\frac{1}{x}-\frac{1}{xe^x}}$$

$$= 1$$

d) $$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{\sin x - x\cos x} \underset{\text{l'Hôpital}}{=} \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{\cos x - \cos x + x\sin x}$$

$$\underset{\text{l'Hôpital}}{=} \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{\sin x + x\cos x} \underset{\text{l'Hôpital}}{=} \frac{-2\cos x + 8\cos 2x}{\cos x + \cos x - x\sin x} = \frac{6}{2} = 3$$

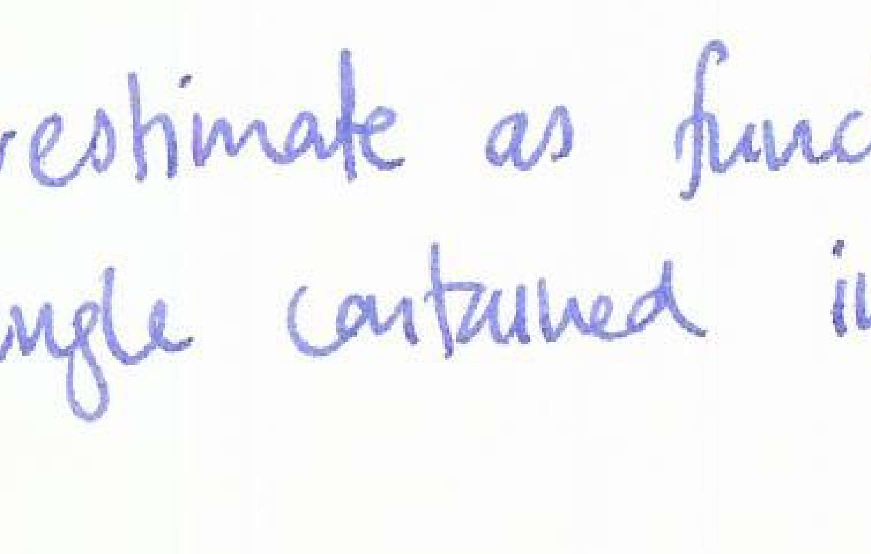
Q8 a) $$\int x + 1 + \frac{1}{x}\, dx = \frac{1}{2}x^2 + x + \ln|x| + c$$

b) $$\int e^x - 4\sin(x)\, dx = e^x + 4\cos(x) + c$$

c) $$\int_1^2 2\sqrt{x}\, dx = \left[2\,\frac{2x^{3/2}}{3} \right]_1^2 = \frac{4}{3}\left(2^{3/2} - 1\right)$$

d) $$\int_0^t \frac{1}{x+1}\, dx = \Big[\ln|x+1| \Big]_0^t = \ln|t+1|$$

Q9.



$$R_4 = \frac{1}{2}\left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2) \right)$$

$$= \frac{1}{2}\left(e^{-1/2} + e^{-1} + e^{-3/2} + e^{-2} \right) \simeq 0.666$$

underestimate as function decreasing so right endpoints give rectangle contained in area under curve.