Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Spring 2013

Instructions: Do at least 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers.

Part I

1. Define *compactness* and *sequential compactness* for a topological space X. If X is a metric space then these two notions coincide; prove one of these two implications.

2. Prove that every countable metric space M containing at least two points, is disconnected. Prove that there exists countable topological spaces with more than two elements, which are connected.

3. Let X be a compact topological space, let Y be a Hausdorff topological space, and let $f\colon X \to Y$ be a continuous, onto map. Show that f is a quotient map, i.e. $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

Part II

1. Let T denote the solid torus (i.e., the product of the two dimensional disk and a circle). Attach two disjoint copies of T, denoted T_1 and T_2 together via a homeomorphism, $\phi\colon \partial T_1 \to \partial T_2$; call the resulting space X_ϕ. Prove or disprove the following claim: for any two homeomorphisms $\phi\colon \partial T_1 \to \partial T_2$ and $\rho\colon \partial T_1 \to \partial T_2$ the resulting spaces X_ϕ and X_ρ are homeomorphic to each other.

2. Let M and N be closed, connected and orientable n-manifolds. Let f : $M \to N$ be a degree one map. Show that $f_*\colon \pi_1(M) \to \pi_1(N)$ is onto.

3. Consider $\mathbb{R}^3$ with both the z-axis removed and the unit circle in the xy-plane removed. Calculate the fundamental group of this space.

4. Find a space whose fundamental group is the free group on n generators. Using the theory of covering spaces, show that every subgroup of the free group on n generators is free.

5. State the classification of closed orientable surfaces. Compute the Euler characteristic of the surface obtained by identifying opposite sides of a hexagon with opposite orientation (i.e. according to the identifications $abcABC$), and identify which surface it is.

Part III

1. Let X and Y be topological spaces for which $H_*(X) = H_*(Y)$. Does it follow that X and Y are homeomorphic? Prove or disprove.

2. Let X be a topological space, and let I be the unit interval $[0, 1]$. Let $CX = X \times I/(X \times \{1\})$ be the cone of X, and let $SX = CX/(X \times \{0\})$ be the suspension of X.

   (a) Prove that CX is contractible.

   (b) Prove that $H_k(X) \cong H_{k+1}(SX)$.

3. Let S and S' be closed orientable surfaces. A map $f : S \to S'$ induces a map on second homology $f_* : H_2(S) \to H_2(S')$. As $H_2(S) \cong H_2(S') \cong \mathbb{Z}$, we can think of this map as multiplication by an integer d, known as the degree of f.

   (a) Construct maps of degree zero, one and two from a closed orientable genus 3 surface to a closed orientable genus 2 surface.

   (b) Show that there is no degree one map from the closed orientable genus 2 surface to the closed orientable genus 3 surface. Hint: think about cup products or use covering spaces.

4. Show that the homology of a closed orientable 3-manifold with trivial first homology is the same as the homology of S^3.

5. Use the Mayer-Vietoris theorem to compute the homology of $S^1 \times S^2$.