Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Fall 2012

Instructions: Do at least 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers.

Part I

1. (a) Define what it means for a topological space to be *regular*
   (b) Prove that a compact Hausdorff space is regular.
   (c) Give an example of a Hausdorff space which is not regular. (No proof required.)

2. (a) Let U be a connected non-empty open subset of $\mathbb{R}^2$. Show that U is path connected.
   (b) Show that every path connected space is connected.

3. Let X be a topological space and $A \subset X$ be a subspace. A is a retract of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Let $A \subset X$ be a retract of X.
   (a) Show that if X is contractible then A is contractible.
   (b) Show that if X is path connected then A is path connected.

4. Prove one of the following:
   (a) (Lebesgue covering lemma) In a sequentially compact metric space, every open cover has a Lebesgue number.
   (b) The product of any non-empty class of connected spaces is connected.

Part II

1. Let M be the surface obtained by identify the edges of an octagon using the pattern $abcbdc^{-1}da^{-1}$.
   (a) Compute the Euler characteristic of M, determine the orientability of M and use it to identify M.
   (b) Compute $\pi_1(M - p)$, where p is the center of the octagon. (Hint: Feel free to consider a homotopy equivalent space).

2. Let X be $(S^1 \times S^1) \vee S^1$, i.e. the one point union (or wedge product) of the torus $S^1 \times S^1$ with the circle S^1. Find all connected 2-fold covers of X. Carefully justify your answer.[HINT: Find homomorphisms onto an appropriate group.]

3. A torsion group is a group G in which every element has finite order; i.e. for each $g \in G$ there exists an integer n such that g^n equals the identity element of G. Suppose Y is a path-connected, locally path-connected space such that $\pi_1(Y)$ is a torsion group. Prove that every map from Y to S^1 is null-homotopic.

4. (a) State the classification theorem for closed connected 2-manifolds.
   (b) Sketch a proof that no two of your spaces are homeomorphic to one another.

5. (a) Show that every map from S^2 to S^1 is null-homotopic.
   (b) Suppose U and V are open subsets covering a space X with $U \cap V$ path connected and $x_0 \in U \cap V$. Suppose $i_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ is surjective. Prove that $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ is surjective.

6. Let α_1 be an essential curve on a two-dimensional torus T_1, e.g. $S^1 \times \{x\} \subset S^1 \times S^1$, and let α_2 be an essential curve on a 2-dimensional torus T_2. Let X be the space formed from the union of T_1 and T_2 by identifying α_1 with α_2 by a homeomorphism. Compute $\pi_1 X$ using Seifert-Van-Kampen Theorem.

Part III

1. Let M be a closed connected orientable n-manifold. Show that $H_{n-1}(M; \mathbb{Z})$ is free abelian. Give an example of a closed orientable manifold with torsion in homology, and verify your example.

2. Use Poincare duality to show that any odd dimensional closed manifold has zero Euler characteristics (Hint: Treat the non-orientable case differently).

3. Let $M = M_1 \# M_2$, the connected sum of two closed n-dimensional manifolds M_1 and M_2. If M_1 is not orientable then prove that M is not orientable.[HINT: You may use the following two facts:

   - The inclusion of the $n-1$ skeleton of an n manifold minus the interior of a small n disk is a homotopy equivalence.

     and

   - The inclusion $S^{n-1} \to M \setminus Int(D^n)$ is an injection if and only if M is not orientable.]

4. Let X be the subset of $\mathbb{R}^3$ consisting of the union of the spheres $A = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : \{(x-1)^2 + y^2 + z^2 = 1\}$. Compute $H_*(X)$.

5. Let X be a space such that

$$H_q(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}\oplus\mathbb{Z}_6 & q=1 \\ \mathbb{Z}_2\oplus\mathbb{Z}_3 & q=2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute $H^q(X;\mathbb{Z})$.

   (b) Compute $H^q(X;\mathbb{Z}_4)$.

6. Let T be the torus $S^1 \times S^1$, and let X be the one point union of two copies of S^1 and S^2, i.e. $S^1 \vee S^1 \vee S^2$. Show that T and X have the same homology groups, and describe the cup product ring structure in each case. Deduce that they are not homotopy equivalent.