Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Spring 2012

Instructions: Do at least 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers.

Part I

1. (a) Prove that X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.

   (b) Show that every retract A of a Hausdorff space X is closed in X. Hint: Show that $X \setminus A$ is open.

2. Prove either the Baire category theorem or Tychonoff's theorem.

3. (a) Show that the product of two connected spaces is connected.

   (b) Let X be a compact metric space. Show that every sequence has a convergent subsequence.

4. (a) If A is a subset of a topological space, X and $f : A \to Y$ a continuous map. We say f can be extended to X if there is a continuous map, $g : X \to Y$ with $g = f$ on A. Prove that if A is dense in X and Y is Hausdorff, then f can be extended to X in at most one way.

   (b) Give an example of spaces X and Y, a dense subset, A and a map $f : A \to Y$ such that f can be extended to X in more than one way.

   (c) Give an example of spaces X and Y, a dense subset, A and a map $f : A \to Y$ such that f cannot be extended.

Part II

1. Let X be the subset of $\mathbb{R}^3$ consisting of the unit sphere, the segment of the z-axis inside the unit sphere, and the unit disc in the xy-plane. Find the fundamental group of X. (Hint: feel free to consider a homotopy equivalent space.)

2. Find all connected 3-fold covers of the wedge sum of a circle and a projective plane, carefully justifying your answer.

3. Prove that if X is a simply-connected, locally path-connected pointed space (i.e. has a distinguished base point) and $p : E \to B$ is a covering projection of pointed spaces, then $p_*[X, E] \to [X, B]$ is bijective. Here $[X, Y]$ denotes the set of pointed homotopy classes of pointed maps from X to Y.

4. Let X be a closed orientable surface and $p : Y \to X$ be a covering map. If Y is homeomorphic to X and p is not a homeomorphism then show that X is a torus.

5. Let M and N be closed, connected and orientable n-manifolds. Let $f : M \to N$ be a degree one map. Show that $f_* : \pi_1(M) \to \pi_1(N)$ is onto.

6. Let X and Y be path connected spaces. Let CX denote the cone of X. The *join* of X and Y is defined as $X * Y = (CX \times Y) \cup_{X \times Y} (X \times CY)$. Compute $\pi_1(X * Y)$ using the Seifert-Van-Kampen theorem.

Part III

1. Let $X = X_1 \cup X_2$, where X_1 and X_2 are both tori $S^1 \times S^1$, and $X_1 \cap X_2$ is a simple closed curve which bounds a disc in X_1 and is essential in X_2. Use the Mayer-Vietoris sequence to find the homology of X.

2. Let S_g be the closed orientable surface of genus g, and let $f : S_2 \to S_3$ be a continuous map. Write down the ring structure on the cohomology of S_2 and S_3, no proof is required. Show that $f^* : H^2(S_3) \to H^2(S_2)$ must be zero, by any method. (Hint: the intersection form is non-degenerate.)

3. Show that $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$ have the same fundamental groups and homology groups. Are these spaces homotopy equivalent? Justify your answer.

4. Let M be an n dimensional manifold. $D^n \subset M$ an n disk embedded in a locally Euclidean neighborhood. Let $\overline{M} = M - D^n$ and $f : S \to \overline{M}$ the inclusion of the $n-1$ sphere into $\partial\overline{M}$.

   (a) Prove that $H_{n-1}(f) = 0$ if M is orientable.

   (b) Prove that $H_{n-1}(f)$ is a injection if M is not orientable.

5. Use the Mayer-Vietoris sequence to compute the homology groups of $S^3 - K$ where K is a knot in S^3 i.e. an embedding of S^1 in S^3.

6. Describe a CW structure on $\mathbb{R}P^n$ and use it to compute $H_*(X; \mathbb{Z}_2)$.