Topology Qualifying Exam

Mathematics Program CUNY Graduate Center

Sept 1st 2011

Instructions: Do at least 8 problems in all, with exactly two problems from Part I and at least two problems from Parts II & III each. Please justify your answers.

Part I

1. (a) Show that a compact subset of a Hausdorff space is closed.

   (b) Show that a contractible space is path connected.

2. (a) Let $\mathrm{int}(A)$ denote the interior of a set $A \subset X$. Show that $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.

   (b) Prove that a finite product of connected spaces is connected.

3. (a) Let $A \subset B \subset \overline{A}$. Show that if A is connected then B is connected.

   (b) Let $f : X \to Y$ be a bijective continuous function. Show that if X is compact and Y is Hausdorff, then f is a homeomorphism.

4. (a) Let A and B be disjoint compact subsets of a Hausdorff space X. Show that there exists disjoint sets U and V containing A and B, respectively.

   (b) Prove that a finite product of locally path-connected spaces is locally path-connected.

Part II

1. 15 (a) Let X and Y be topological spaces with basepoints $x \in X$ and $y \in Y$ respectively. Prove that $\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$.

   5 (b) Compute $\pi_1(K \times T^2, (x, y))$ for $n \geq 2$, where K is the Klein's bottle and T^2 is the torus.

2. 10 (a) Classify all covering spaces of $\mathbb{RP}^2 \times \mathbb{RP}^4$.

   10 (b) Prove that every map $\mathbb{RP}^2 \to T^2$ is null homotopic.

3. 20 Let M_g denote the closed orientable surface of genus g. Prove that M_p covers M_q if and only if $p = n(q - 1) + 1$ where n is the number of sheets in the covering.

4. Let M_g denote the closed orientable surface of genus g.

   15 (a) Compute $\pi_1(M_g, x)$ for some appropriate base-point $x \in M_g$.

   5 (b) Prove that M_g is homeomorphic to M_h if and only if $g = h$.

5. Let X be a CW complex with one 0-cell, three 1-cells a, b, c and two 2-cells e and f, with attaching maps given by the words ab^{-2} and cab^2 respectively.

   15 (a) Compute $\pi_1(X, x)$ with some appropriate base-point x.

   5 (b) Using the fundamental group to compute $H_1(X; \mathbb{Z})$.

6. 15 (a) Let Y be a finite CW complex and $p : X \to Y$ be an n-sheeted covering space. Show that $\chi(X) = n\,\chi(Y)$.

   5 (b) Let Y be a finite CW complex and $p : \mathbb{RP}^{2n} \to Y$ be a covering map. Prove that f is a homeomorphism.

Please Turn Over

Part III

1. Let X be a closed surface given by the identification of sides of an octagon using the word $cdbabc^{-1}a^{-1}d$.

   5 (a) Describe the CW structure on X given by this description.

   10 (b) Compute the cellular chain complex and use it to compute $H_*(X;\mathbb{Z})$.

   5 (c) Identify the surface.

2. Compute $H_*(\mathbb{R}\mathrm{P}^5;\mathbb{Z})$ and $H_*(\mathbb{R}\mathrm{P}^5;\mathbb{Z}_2)$.
   10 5

3. Let $X = S^1 \vee S^1 \vee S^2$ and $Y = T^2$.

   10 (a) Compute the homology and cohomology groups of X and Y with $\mathbb{Z}$ coefficients. Are they the same ?

   5 (b) Compute the cohomology rings of X and Y.

   5 (c) Are the spaces homeomorphic ? Are the spaces homotopy equivalent ? Explain.

4. 10 (a) Let M be a closed n–manifold and let $x \in M$. Prove that $H_n(M, M - \{x\}; R) = R$.

   10 (b) Let M be a closed, orientable n–manifold. Prove that $H_{n-1}(M;\mathbb{Z})$ is free. You can assume that the homology groups of M are finitely generated.

5. 20 Use Mayer-Vietoris sequence to compute the homology groups of $S^3 - K$ where K is a knot i.e. an embedding of S^1 in S^3.

6. 20 Use Poincare duality to show that any odd dimensional manifold has zero Euler characteristics (Hint: Treat the non-orientable case differently).